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A STOCHASTIC MINIMUM PRINCIPLE

ROBERT J. ELLIOTT

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1. Pontrjagin's maximum principle [6] is a basic result in deterministic optimal control theory. Analogous results have been obtained for the optimal control of stochastic dynamical systems (see for example the survey by Fleming [3]), and a new approach to such problems, using the martingale theory of Meyer, was made in the paper of Davis and Varaiya [2]. In this paper, by observing that the cost function is a 'semimartingale speciale' (see [5]), we are able to simplify much of [2] and obtain quickly a very general dynamic programming minimum principle.

2. The dynamics are described by a stochastic differential equation

$$dx_t = f(t, x, u)dt + \sigma(t, x)dB_t \tag{2.1}$$

with initial condition $x(0) = x_0 \in R^m$. Here $t \in [0, 1]$, B is an m-dimensional Brownian motion and $x \in C$, the space of continuous functions from $[0, 1]$ to R^m. Write $F_t = \sigma\{x_s: s \leqslant t\}$ for the σ-field generated on C up to time t. The control values u are chosen from a compact metric space U. We suppose f and (nonsingular) σ satisfy the usual measurability and growth conditions (see [2]).

If an m-dimensional Brownian motion B_t on a probability space (Ω, μ) is given these conditions on σ ensure that the equation

$$x_t = x_0 + \int_0^t \sigma(s, x)dB_s$$

has a unique solution with sample paths in C.

DEFINITION 2.1. The admissible controls M_s^t over $[s, t] \subset [0, 1]$ are the measurable functions u: $[s, t] \times C \longrightarrow U$ (U is given the Borel σ-field) such that (i) for each τ, $s \leqslant \tau \leqslant t$, $u(\tau, \cdot)$ is F_τ measurable, (ii) for each $x \in C$, $u(\cdot, x)$ is Lebesgue measurable.

Such functions are nonanticipative feedback controls and the conditions on f ensure that for such a control $u(t, x) \in M_s^t$: $E[\exp[\xi_s^t(f^u)] F_s] = 1$ a.s. μ. Here $f^u(\tau, x) = f(\tau, x, u(\tau, x))$ and

$$\xi_s^t(f^u) = \int_s^t \{\sigma^{-1}(\tau, x)f^u(\tau, x)\}' dB_\tau - \frac{1}{2}\int_s^t |\sigma^{-1}(\tau, x)f^u(\tau, x)|^2 \, d\tau.$$

Writing $M = M_0^1$, for each $u \in M$ a measure μ_u is defined on (C, F_1) by

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putting $d\mu_u/d\mu = \exp \xi_0^1(f^u)$. Girsanov's Theorem ([4], [2]) then states the following:

THEOREM 2.2. *Under the measure μ_u the process w_t^u is a Brownian motion on Ω, where $dw_t^u = \sigma^{-1}(t, x)(dx_t - f^u(t, x)dt$.*

3. The cost associated with this process is supposed to be of the form

$$g(x(1)) + \int_0^1 c(t, x, u)\,dt$$

where g and c are real and bounded, $g(x(1))$ is F_1 measurable and c satisfies the same conditions as the components of f. Corresponding to a control $u \in M$ the expected total cost is

$$J(u) = E_u\left[g(x(1)) + \int_0^1 c_t^u\,dt\right],$$

where $c_t^u = c(t, x, u(t, x))$ and E_u denotes expectation with respect to μ_u. The optimal control problem is to determine how $u \in M$ should be chosen so that $J(u)$ is minimized. The minimum cost that can be incurred from time t onwards is independent of the control used up to time t and is

$$W(t) = \bigwedge_{v \in M_t^1} E_v\left[g(x(1)) + \int_t^1 c_t^v\,dt | F_t\right].$$

Here $\bigwedge$ denotes the infimum in the complete ordered lattice $L^1(\Omega, \mu)$. From the 'principle of optimality' [2, Theorem 3.1] we have:

THEOREM 3.1 (i) *$u^* \in M$ is an optimal control if and only if $W(t) + \int_0^t c_s^{u^*}\,ds$ is a martingale on (Ω, μ_{u^*})*, (ii) *in general, for $u \in M$ $W(t) + \int_0^t c_s^u ds$ is a submartingale on (Ω, μ_u).*

4. From the martingale representation results of Clark [1] we can conclude that $u^* \in M$ is optimal if and only if there is a predictable process g_t^* such that $\int_0^1 |g_s^*|^2\,ds < \infty$ a.s. and $W(t) + \int_0^t c_s^{u^*}\,ds = J^* + \int_0^t g^*\,dw^{u^*}$.

Here $J^* = W(0)$ and the last integral is a stochastic integral with respect to the Brownian motion w^{u^*} on (Ω, μ_{u^*}). In general, for $u \in M$ the submartingale $W(t) + \int_0^t c_s^u\,ds$ has a unique Doob-Meyer decomposition as $J^* + M_t^u + A_t^u$, where M_t^u is a martingale on (Ω, μ_u) and A_t^u is a predictable increasing process. However:

$$\begin{aligned} W(t) + \int_0^t c_s^u\,ds &= J^* + \int_0^t g^* dw^{u^*} + \int_0^t (c_s^u - c_s^{u^*})\,ds \\ &= J^* + \int_0^t g^*\sigma^{-1}(dx_s - f_s^u\,ds) + \int_0^t (g^*\sigma^{-1} f_s^u + c_s^u) - (g^*\sigma^{-1} f_s^{u^*} + c_s^{u^*})\,ds. \end{aligned}$$

From Theorem 2.2, $dw_s^u = \sigma^{-1}(dx_s - f_s^u ds)$ is a Brownian motion on (Ω, μ_u) and so $\int g^* dw^u$ is (a priori, local—but in fact a) martingale. Also, $\int_0^t (g^*\sigma^{-1} f^u + c^u) - (g^*\sigma^{-1} f^{u^*} + c^{u^*})\,ds$ is a predictable process. Because the submartingale $W(t) + \int_0^t c_s^u\, ds$ is a 'semimartingale special' (see Meyer [5]) its decomposition into a constant plus a martingale plus a predictable integrable variation process is unique. Therefore, in the Doob-Meyer decomposition we must have

$$M_t^u = \int_0^t g^* dw^u, \quad A_t^u = \int_0^t (g^*\sigma^{-1} f^u + c^u) - (g^*\sigma^{-1} f^{u^*} + c^{u^*})\, ds.$$

Because A_t^0 is monotonic increasing we have immediately the following principle of optimality:

THEOREM 4.1. *Almost surely (Lebesgue $\times\ \mu$) for every $(t, x) \in [0,1] \times C$*

$$g^*\sigma^{-1} f^{u^*} + c^{u^*} = \inf_{u \in U} g^*\sigma^{-1} f^u + c^u.$$

That is, the optimum control is obtained by minimizing the Hamiltonian $g^*\sigma^{-1} f^u + c^u$. If there is not an optimal control, one can consider a sequence of controls approximating the infimum and use compactness of $\mathcal{D}(\Phi)$ as in Davis and Varaiya [2]. There is a similar result for the partially observable case; however, the techniques are more complicated and will be reported elsewhere.

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DEPARTMENT OF PURE MATHEMATICS, UNIVERSITY OF HULL, HULL, ENGLAND