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GLOBAL RESULTS IN CONTROL THEORY WITH APPLICATIONS TO UNIVALENT FUNCTIONS

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1. A problem in control theory. Many classical coefficient problems in the theory of univalent functions can be stated as the following control problem. Consider a first order differential system

$$dx/dt = f(x, u(t)),$$

where $x = (x_1, \ldots, x_n)$, $u = (u_1, \ldots, u_m)$ and $f(x, u) = (f_1(x, u), \ldots, f_n(x, u))$ are real valued vectors. Assume that f is continuous on $R^n \times R^m$ and for fixed u, $f \in C^1(R^n)$. The values of $u(t)$ are in a compact domain $\mathcal{U} \subset R^m$. Denote by $\mathcal{F}$ the class of all piecewise continuous functions $u(t)$ for $t \geqslant 0$ with the values in $\mathcal{U}$. Let $x(t)$ satisfy a fixed initial condition $x(0) = \xi$. Denote by $x(t, u)$ the solution of the system above for a given $u(t)$ in $\mathcal{F}$. Let $F(x) = F(x_1, \ldots, x_n)$ belong to $C^1(R^n)$.

THEOREM 1. *Let $u^* = u^*(t)$ be a solution of the problem* $\sup_{\mathcal{F}} F(x(T, u)) = F(x(T, u^*))$, *for $T > 0$. Consider the system*

$$dx/dt = f(x, u^*(t)), \; x(\tau) = \eta$$

for $0 \leqslant \tau \leqslant T$. Define a function F_τ by the equality $F_\tau(\eta) = F(x(T))$. Then $x(\tau, u^*)$ solves the problem $\sup_{\mathcal{F}} F_\tau(x(\tau, u)) = F_\tau(x(\tau, u^*))$.

The proof of the theorem follows by considering the functions $u(t)$ such that $u(t) = u^*(t)$ for $\tau < t \leqslant T$. In case where $f(x, u) = A(u)x$ and $F(x) = \lambda_0' x$ Theorem 1 has a very simple form. Here $A(u) = (a_{ij}(u))_1^n$ and $a_{ij}(u) \in C(R^m)$. By A' and λ' we denote the corresponding transposed matrix and vector.

THEOREM 2. *Consider a control system $dx/dt = A(u(t))x$. Let $u^*(t)$ solve the linear problem*

$$\sup_{\mathcal{F}} \lambda_0' x(T, u) = \lambda_0' x(T, u^*).$$

Then $x(\tau, u^*)$ solves the linear problem

$$\sup_{\mathcal{F}} \lambda'(\tau) x(\tau, u) = \lambda'(\tau) x(\tau, u^*),$$

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$0 < \tau < T$, *where* $\lambda(t)$ *is the solution of* $d\lambda/dt = -A'(u^*)\lambda$, $\lambda(T) = \lambda_0$.

2. Univalent functions. Denote by S the set of all analytic univalent functions $f(z) = z + \Sigma_{k=2}^{\infty} a_k z^k$ in the unit disc D. Let f be a slit function, i.e. f maps D onto a slit domain. According to Loewner [1] f can be embedded in a semigroup of univalent functions $g(z, t) = e^t(z + \Sigma_{k=2}^{\infty} a_k(t) z^k)$ which satisfies the equation

$$\frac{\partial g}{\partial t} = z \frac{\partial g}{\partial z} \frac{1 + e^{i\varphi(t)}z}{1 - e^{i\varphi(t)}z}, \qquad g(z, 0) = f(z).$$

Here $\varphi(t)$ is a real piecewise continuous function for $t \geqslant 0$. Denote by A_n the set of vectors $a^{(n)} = (a_1, \ldots, a_n)$, $(a_1 = 1)$ which are the first n coefficients of some f in S. Let $f(z, t) = e^{-t}g(z, t) = z + \Sigma_{k=2}^{\infty} a_k(t) z^k$. Then $a^{(n)}(t)$ satisfies the system

$$da^{(n)}(t)/dt = e^{i\varphi(t)G_n} A_n e^{-i\varphi(t)G_n} a^{(n)}(t), \qquad a^{(n)}(0) = a^{(n)}.$$

Here $A_n = (a_{kj})_1^n$ and $G_n = (d_k \delta_{kj})_1^n$ are the matrices: $a_{kj} = 0$ for $j > k$, $a_{kk} = k - 1$, $a_{kj} = 2j$ for $j < k$, $d_k = k - 1$, $k, j = 1, \ldots, n$. The following result is basic for applications of the method of control theory to coefficient problems for univalent functions.

THEOREM 3. *Let* $\varphi(t)$ *be a real measurable function for* $t \geqslant 0$. *Consider the system* $da^{(n)}/dt = -e^{i\varphi(t)G_n} A_n e^{-i\varphi(t)G_n} a^{(n)}$ *for* $t \geqslant 0$.

Then (i) A_n *is invariant under the flow defined by the system above. That is, if* $a^{(n)}(0) \in \mathsf{A}_n$ *then* $a^{(n)}(t) \in \mathsf{A}_n$ *for any* $t > 0$.

(ii) *Let* $\alpha^{(n)} \in \mathsf{A}_n$ *and consider the set of all possible paths* $a^{(n)}(t)$ *starting from the point* $\alpha^{(n)}$ *for all choices of* φ. *Then this set is dense in* A_n.

Let $a_*^{(n)} = (a_1^*, \ldots, a_n^*)$ be a boundary point of A_n. According to [2], the corresponding function $f^*(z) = z + \Sigma_{k=2}^{\infty} a_k^* z^k$ is a slit function. So $f^*(z)$ generates the corresponding $\varphi^*(t)$ which appears in the Loewner equation. Using Theorems 2 and 3, we obtain

THEOREM 4. *Let* $a_*^{(n)}$ *solve the problem*

$$\max_{\mathsf{A}_n} \operatorname{Re}\left\{\sum_{k=1}^{n} \lambda_k^0 a_k\right\} = \operatorname{Re}\left\{\sum_{k=1}^{n} \lambda_k^0 a_k^*\right\}$$

subject to m *constraints* $a_k = a_k^*$, $k = 1, \ldots, m$, $(m \leqslant n - 1)$. *Let* $a^{(n)}(t)$ *be generated by the Loewner equation*

$$da^{(n)}/dt = e^{i\varphi^*(t)G_n} A_n e^{-i\varphi^*(t)G_n} a^{(n)}, \qquad a^{(n)}(0) = a_*^{(n)}.$$

Define $\lambda^{(n)}(t)$ *to be*

$$d\lambda^{(n)}/dt = -e^{-i\varphi^*(t)G_n} A'_n e^{i\varphi^*(t)G_n} \lambda^{(n)}, \qquad \lambda^{(n)}(0) = \lambda_0^{(n)}.$$

Then $a^{(n)}(t)$ *solves the problem*

$$\max_{A_n} \operatorname{Re}\left\{\sum_{k=1}^{n} \lambda_k(t) a_k\right\} = \operatorname{Re}\left\{\sum_{k=1}^{n} \lambda_k(t) a_k(t)\right\}$$

$$a_k = a_k(t),\ k = 1, \ldots, m, \text{ for } t > 0.$$

In particular we obtain that if $a_*^{(n)}$ is a supporting point of A_n, so is $a^{(n)}(t)$ for $t > 0$ [2, 10.3]. Let $\xi^k = (0, \ldots, 0, 1, \xi^k_{k+2}, \ldots, \xi^k_n)$ and $\eta^k = (\eta^k_1, \ldots, \eta^k_k, 1, 0, \ldots, 0)$, $k = 0, \ldots, n-1$, where

$$\xi^k_r = (-1)^k \binom{r+k}{r-k-1}, \eta^k_r = (-1)^{r-1} \frac{r}{k+1} \binom{2k+2}{k-r+1}.$$

THEOERM 5. *Assume that the Koebe function* $K(z) = z/(1-z)^2$ *solves a linear problem*

$$\max_{A_n} \operatorname{Re}\{\lambda^{(n)\prime} a^{(n)}\} = \operatorname{Re}\{\lambda^{(n)\prime} e^{(n)}\}$$

where $e^{(n)} = (1, 2, \ldots, n)$.

Then the Koebe function also satisfies $\max_{A_n} \operatorname{Re}\{\lambda^{(n)}(x)' a^{(n)}\} = \operatorname{Re}\{\lambda^{(n)}(x)' e^{(n)}\}$, *for* $0 < x < 1$, *where* $\lambda^{(n)}(x) = \Sigma_{r=0}^{n-1} x^r (\lambda^{(n)\prime} \xi^r) \eta^r$. *In particular, if* $\operatorname{Re}\{a_n\} \leqslant n$, *then*

$$\operatorname{Re}\left\{\sum_{r=0}^{n-1} (-1)^r x^r \binom{n+r}{n-r-1} [(\eta^r)' a^{(n)}]\right\}$$

$$\leqslant \operatorname{Re}\left\{\sum_{r=0}^{n-1} (-1)^r x^r \binom{n+r}{n-r-1} [(\eta^r)' e^{(n)}]\right\}, \quad \text{for } 0 < x < 1.$$

Thus, from $\operatorname{Re}\{a_4\} \leqslant 4$ we obtain the inequality

$$\operatorname{Re}\{x^2 a_4 + 6x(1-x) a_3 + 2(7x^2 - 12x + 5) a_2\}$$
$$\leqslant 14x^2 - 30x + 20$$

for $0 < x < 1$. The full details and the proofs will appear elsewhere.

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