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APPLICATIONS OF THE INTEGRAL OF AN INVARIANT OF THE HESSIAN

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In this paper we announce several results whose proofs will appear elsewhere.

Throughout this paper M will be an n-dimensional smooth, connected, compact and oriented Riemannian manifold with boundary N.

DEFINITION. If f is a smooth function on M then the rth invariant, $S_r(f)$, of the Hessian operator of f, where $r = 0, 1, \ldots, n$, is the rth elementary symmetric function of the eigenvalues of this operator.

THEOREM. 1. *If M, N and f are as above then*

$$\int_M 2S_2(f)\Omega = \int_N \{(\Delta z - uK_1)u - \langle \nabla z, \nabla u\rangle - \mathrm{II}(\nabla z, \nabla u)\}\Psi + \int_M \mathrm{Ric}(\operatorname{grad} f, \operatorname{grad} f)\,\Omega. \tag{1}$$

In this formula $z = f|_N$, u is the exterior derivative of f along N, K_1 is $n - 1$ times the mean curvature and II is the second fundamental form (of N in M), while Δ is the Laplace operator, ∇ is the gradient and Ψ is the volume form on N; similarly, grad is the gradient, Ric is the Ricci tensor and Ω is the volume form on M.

REMARK. We have a similar formula involving $S_r(f)$ for $r \geqslant 3$.

THEOREM 2. *Let M, N be as above and assume that* (a) *for some constant $c^2 > 0$ all the Ricci curvatures on M are bounded below by $(n-1)c^2$ and* (b) *$K_1 \leqslant 0$. Then the first eigenvalue λ_1 for the Laplace operator on M (for the fixed membrane problem) satisfies the inequality $\lambda_1 \geqslant nc^2$. Moreover, equality occurs if and only if M is isometric to a full hemisphere of the standard sphere of radius $1/c$.*

REMARK. This generalizes to the situation of manifolds with boundary the well-known theorems of Lichnerowicz [3] and Obata [4]. Our proof uses Theorem 1 and an analog of Obata's Theorem A.

COROLLARY. *Let N be a compact minimally imbedded hypersurface in the n-sphere of radius $1/c$ and let M be one of the two domains into which N cuts the sphere. If λ_1 is the first eigenvalue of the Laplace operator on M, then $\lambda_1 \geqslant nc^2$, with equality if and only if N is imbedded as an equator.*

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Theorem 1 also provides a simple proof of the following well-known theorem of Aleksandrov [1].

THEOREM 3. *If N is a compact hypersurface in* $\mathbf{R}^n$, *with constant mean curvature, which bounds a compact domain M, then N is a hypersphere.*

In our proof of Theorem 3 we apply Theorem 1 to a solution of the Poisson equation $\Delta f = 1$, with boundary condition $f = 0$ on N.

Our final application of Theorem 1 extends results of Flanders [2].

THEOREM 4. *Let W be an n-dimensional Riemannian manifold of positive-semidefinite Ricci curvature. For a point p in W and real $t > 0$ let $B(p, t)$ and $S(p, t)$ denote (respectively) the closed ball and the sphere of radius t about p. Suppose that $T > 0$ is small enough that for all $t \in (0, T]$, $B(p, t)$ is a normal neighborhood of p and $S(p, t)$ has negative semidefinite second fundamental form (relative to the exterior normal to $B(p, t)$). Denote the volume of $B(p, t)$ by $V(t)$. If f is a smooth function which satisfies on $B(p, t)$ an inequality $S_2(f) \leqslant -\alpha(1 + |\operatorname{grad} f|^2)^{1+2\epsilon}$ for certain positive constants α and ϵ, then for each $t_1 \in (0, T]$ we have*

$$1 \leqslant \epsilon \left(\frac{2\alpha}{1+\epsilon}\right)^{1/2} \int_{t_1}^{T} \left(\frac{V(t_1)}{V(t)}\right)^{\epsilon} dt. \tag{2}$$

Our proof uses Theorem 1 (with $M = B(p, T)$) together with a generalization of formula 3.1 of [2].

REMARK. Flanders proves (2) in the special case $W = \mathbf{R}^n$ and uses it to estimate (in terms of α, ϵ and n) an upper bound on T. We are able to obtain similar bounds in certain other cases, for example:

COROLLARY. *Suppose that W is a two-dimensional sphere of radius R. If there is a function f which satisfies the hypotheses of Theorem 4 with $\epsilon = 1$, then* $T \leqslant 4R \arcsin(1/(1 + R^2\alpha)^{1/2})$.

REMARK. We arrive at the estimate obtained by Flanders in the corresponding case $W = R^2$, $\epsilon = 1$, by letting R tend towards infinity.

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