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COBORDISM FOR POINCARÉ DUALITY GROUPS

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1. **Relative homology for pairs.** Homology and cohomology for a pair of groups $G \supset S$ (cf. [6]) can be extended to pairs (G, S) consisting of a group G and a family of subgroups $S = \{S_i\}$, as follows: If $S = \varnothing$ one takes the usual (absolute) groups of G. If $S \neq \varnothing$, let Δ be the kernel of the G-homomorphism $\bigoplus_i \mathbf{Z}(G/S_i) \twoheadrightarrow \mathbf{Z}$ given by augmentations; A being a G-module, we put $H^k(G, S; A) = H^{k-1}(G; \operatorname{Hom}(\Delta, A))$ and $H_k(G, S; A) = H_{k-1}(G; \Delta \otimes A)$ where G acts diagonally in $\operatorname{Hom}(\Delta, A)$ and $\Delta \otimes A$. One has exact sequences

$$\cdots \longrightarrow H^k(G, S; A) \longrightarrow H^k(G; A) \longrightarrow \prod_i H^k(S_i; A) \xrightarrow{\delta} H^{k+1}(G, S; A) \longrightarrow \cdots,$$

$$\cdots \longrightarrow H_{k+1}(G, S; A) \xrightarrow{\partial} \bigoplus_i H_k(S_i; A) \longrightarrow H_k(G; A) \longrightarrow H_k(G, S; A) \longrightarrow \cdots.$$

2. **Poincaré duality pairs.** The product structure for Ext^* and Tor_* (cf. [4, Chapter XI]) yields, for $\alpha \in H_n(G, S; B)$, cap-products $\alpha \cap -: H^k(G; A) \longrightarrow H_{n-k}(G, S; B \otimes A)$ and $H^k(G, S; A) \longrightarrow H_{n-k}(G; B \otimes A)$, with diagonal G-action in $B \otimes A$.

(1) DEFINITION. (G, S) is a *Poincaré duality pair of dimension n* (in short: PD^n-pair) if there is a G-module structure $\widetilde{\mathbf{Z}}$ on $\mathbf{Z}$ and an element $e \in H_n(G, S; \widetilde{\mathbf{Z}})$ such that $e \cap -: H^k(G; A) \longrightarrow H_{n-k}(G, S; \widetilde{A})$ is an isomorphism for all A and k.

$\widetilde{\mathbf{Z}}$ is called the "dualizing" module. $\widetilde{A}$ stands for $\widetilde{\mathbf{Z}} \otimes A$. If S is empty, (1) coincides with the usual definition (cf. [1]) of a *Poincaré duality group* of dimension n (in short: PD^n-group).

(2) THEOREM. *(G, S) is a PD^n-pair with dualizing module $\widetilde{\mathbf{Z}}$ if and only if G is a duality group (cf. [3]) with dualizing module $\widetilde{\Delta}$.*

Thus various results and criteria for duality groups can be applied to PD^n-pairs; note that $\widetilde{\Delta}$ is $\mathbf{Z}$-free. We give here, and in §3, a list of consequences of (1): Definition (1) is equivalent with $e \cap -: H^k(G, S; A) \longrightarrow H_{n-k}(G; \widetilde{A})$ being an isomorphism for all A and k. The module $\widetilde{\mathbf{Z}}$ and the dimension n are determined by the pair (G, S). The pair is called *orientable* if G acts trivially on $\widetilde{\mathbf{Z}}$,

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otherwise *nonorientable.* The group $H_n(G, S; \widetilde{\mathbf{Z}})$ is infinite cyclic generated by e; choice of a generator e is called an orientation (even in the nonorientable case!). The family S is finite; all S_i in S are PD^{n-1}-groups; an orientation of (G, S) determines an orientation for each S_i. We call S (with these orientations) the *boundary* of the (oriented) PD^n-pair (G, S). For an oriented group the minus sign denotes change of orientation.

3. **Extensions and amalgams.** Using (2) and criteria for duality one proves

(3) THEOREM. *Let $N \rightarrowtail G \overset{\pi}{\twoheadrightarrow} Q$ be a short exact sequence of groups, S a family of subgroups S_i of G each containing N, and $K = \pi(S)$. If N is a PD^n-group and (Q, K) is a PD^m-pair, then (G, S) is a PD^{n+m}-pair.*

There is a converse of this in the sense of [2, Theorem A]; and there is also a "finite extension Theorem" generalizing the corresponding result in [1] and [5].

(4) THEOREM. (a) *If the subgroups U of G and V of H are isomorphic, and if $(G, S \cup U)$ and $(H, R \cup V)$ are PD^n-pairs then so is $(G *_U H, S \cup R)$.*[1]

(b) *Let U, V be isomorphic subgroups of G, with $\sigma\colon U \xrightarrow{\cong} V$. If $(G, S \cup U \cup V)$ is a PD^n-pair, then so is $(G *_{U,\sigma}, S)$, where $G *_{U,\sigma}$ is the HNN-extension for σ.*

Repeated application of (4) yields, for the "fundamental group $G(\mathfrak{G})$ of a graph of groups" in the sense of Serre [7]:

(5) THEOREM. *Let (G^ν, S^ν) be a finite family of PD^n-pairs, and $\mathfrak{G}$ a graph of groups whose vertices $V(\mathfrak{G})$ are the groups G_ν and whose edges $E(\mathfrak{G})$ some pairs of isomorphic subgroups $S_i^{(\nu)}, S_{i'}^{\nu'} \in \bigcup_\nu S^{(\nu)}$. Further, let S be the family of those $S_i^{(\nu)}$ not occurring in $E(\mathfrak{G})$. Then $(G(\mathfrak{G}), S)$ is a PD^n-pair.*

4. **Oriented cobordism.** Here all PD-groups and -pairs are assumed orientable and oriented.

(6) DEFINITION. Two finite families S, T of PD^{n-1}-groups are *cobordant* if there exist finitely many PD^n-pairs such that $S \cup (-T)$ is the disjoint union of their boundaries.

This is an equivalence relation: it is easily seen that it is reflexive and symmetric; transitivity is proved by means of (5). We write $\langle S\rangle$ for the class of S, and Ω_n for the set of all classes of families S of PD^n-groups. With addition of classes defined by $\langle S\rangle + \langle T\rangle = \langle S \cup T\rangle$, Ω_n is an Abelian group ($0 = \varnothing$, $-\langle S\rangle = \langle -S\rangle$); it is generated by the classes $\langle S\rangle$ of single groups. $\Omega_0 \cong \mathbf{Z}$, generated by the class $\langle 1\rangle$ of the trivial group. The direct product of groups defines a graded (skew-) commutative multiplication $\Omega_n \times \Omega_m \longrightarrow \Omega_{n+m}$ turning the graded group Ω_* into a graded ring with unit.

[1] For families S, T, the union $S \cup T$ is taken with respect to the *disjoint* union of the indexing sets.

The *signature* of an orientable PD^n-group is defined as for manifolds (for dimensions $4k$, otherwise 0). One can show that it is 0 for $\langle S \rangle = 0$, and that it defines a unitary ring homomorphism $\Omega_* \longrightarrow \mathbf{Z}$. From known examples of PD^4-groups with signature $\neq 0$ (even with arbitrarily large signature) it follows that Ω_* does not consist of Ω_0 only.

Oriented cobordism can, of course, also be defined by admitting orientable *and* nonorientable PD^n-groups and -pairs. Using (4)(a) above, one proves that the parity of the Euler characteristic is an invariant. Nonorientiable higher genus surfaces with odd Euler characteristic thus provide examples of PD^2-groups S with $\langle S \rangle \neq 0$.

5. **Nonoriented cobordism.** If orientable and nonorientable PD^n-groups and -pairs are admitted and all orientations disregarded, the nonoriented cobordism ring $\mathfrak{N}_*$ is obtained (all elements are of order 2). One then can prove that for $\langle S \rangle = 0$ all Stiefel-Whitney numbers of S are 0 (mod 2 characteristic classes for PD^n-groups and -pairs can be defined algebraically via the Steenrod squares and the Wu formula).

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