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STABILITY OF EQUIVARIANT SMOOTH MAPS

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This research announcement is a summary of a paper which will appear elsewhere [5], and which continues the program started in [4].

1. We consider a compact Lie group G and smooth compact G-manifolds X and Y. By $C_G^\infty(X, Y)$, $\mathrm{Diff}_G(X)$, $\mathrm{Diff}_G(Y)$ we denote the C^∞, *G-equivariant* mappings $X \longrightarrow Y$, respectively, diffeos of X or diffeos of Y.

There is a natural group action

$$\mathrm{Diff}_G(X) \times \mathrm{Diff}_G(Y) \times C_G^\infty(X, Y) \xrightarrow[\Phi]{} C_G^\infty(X, Y),$$

and for each $f \in C_G^\infty(X, Y)$, we define the corresponding orbit-map

$$\mathrm{Diff}_G(X) \times \mathrm{Diff}_G(Y) \xrightarrow[\Phi_f]{} C_G^\infty(X, Y).$$

We consider the G-bundles TX, TY, f^*TY and their "invariant sections" $\Gamma^\infty(TX)^G$, $\Gamma^\infty(TY)^G$, $\Gamma^\infty(f^*TY)^G$. (These are modules over the corresponding rings of G-invariant functions.)

As in the usual case [3], [6] we have linear mappings

$$\begin{array}{ccc} \Gamma^\infty(TX)^G & \searrow^{\beta_f} & \\ & & \Gamma^\infty(f^*TY)^G \\ \Gamma^\infty(TY)^G & \nearrow_{\alpha_f} & \end{array}$$

defined in a natural way.

By definition, f is infinitesimally stable if $\alpha_f + \beta_f$ is surjective.

By definition, f is stable if Image Φ_f is a neighbourhood of $f \in C_G^\infty(X, Y)$.

With these definitions we have the

STABILITY THEOREM. *Let $f \in C_G^\infty(X, Y)$ be infinitesimally stable. Then:*

(i) *Whenever Z_1 is the germ of a metrizable or compact topological space, Z_2 the germ of a smooth finite dimensional manifold, and $\psi\colon Z_1 \times Z_2 \longrightarrow C_G^\infty(X, Y)$ a $C^{0,\infty}$-germ of a map sending the base points to f, there is a germ of a $C^{0,\infty}$ map $\Psi\colon Z_1 \times Z_2 \longrightarrow \mathrm{Diff}_G(X) \times \mathrm{Diff}_G(Y)$ sending the base points to* (id X) $\times$ (id Y) *and such that*

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$$\begin{array}{ccc} & & \mathrm{Diff}_G(X) \times \mathrm{Diff}_G(Y) \\ & \overset{\Psi}{\nearrow} & \downarrow \Phi_f \\ Z_1 \times Z_2 & \xrightarrow[\psi]{} & C^\infty_G(X, Y) \end{array}$$

is commutative.

(ii) *There is a neighbourhood* $f \in N \subset C^\infty_G(X, Y)$ *such that every* $f' \in N$ *is also infinitesimally stable*

(iii) f *is stable.* □

The proof relies heavily on the work of J. Mather [2], [3] (which is generalized by this theorem) and of G. Schwarz [7].

2. Let $G \xrightarrow{\psi} \mathrm{Aut}(V)$ be a linear representation of the compact Lie group G. Let $x \in V$ be the current point of V and $R[x]^G \subset R[x]$ be the algebra of G-invariant polynomials. According to a classical theorem of Hilbert [1], [8] we can always choose a *finite* system $(\rho_1, \ldots, \rho_k) = \rho \subset R[x]^G$ of algebra generators of $R[x]^G$. We shall attach to the representation (G, ψ) the number

$$\mathrm{ord}(G, \psi) = \min_\rho \left(\max_i \deg \rho_i \right) \in Z^+.$$

Suppose now X is a (*not* necessarily compact) G-manifold (G compact). By the slice-representation we have a naturally defined function on the space of orbits $X/G \xrightarrow{\mathrm{ord}} Z^+$.

One of the technical ingredients occurring in the context of the stability theorem is the following

SEMICONTINUITY LEMMA. *For every orbit* $\{Gx\} \in X/G$ *there exists a neighbourhood* $\{Gx\} \in W \subset X/G$, *such that, for any* $\{Gx'\} \in W$, *one has*

$$\mathrm{ord}(Gx') \leqslant \mathrm{ord}(Gx). \quad \square$$

This might be useful in the study of deformations of group actions suggested by Thom.

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