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OPERATOR ALGEBRAS AND ALGEBRAIC K-THEORY

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1. **Introduction.** We wish to announce several related results which demonstrate a relationship between operator theory and algebraic K-theory. Some of these results concern extensions of C^*-algebras (cf. [4], [5]) and complement the results of [4]. Others concern the trace and determinant invariants defined in [7].

2. **Extensions of C^*-algebras.** Let H be a separable infinite dimensional Hilbert space, $L(H)$ the algebra of bounded linear operators on H, K the ideal of compact operators, and $A = L(H)/K$. In [4] and [5] $\mathrm{Ext}(X)$ was defined as the set of equivalence classes of C^*-algebra extensions, $0 \to K \to E \to C(X) \to 0$, for X a compact metric space and $C(X)$ the algebra of continuous complex functions on X. $\mathrm{Ext}(X)$ was also described as unitary equivalence classes of $*$-isomorphisms $\tau\colon C(X) \to A$. It was shown that $\mathrm{Ext}(X)$ is a group and that it gives rise to a generalized homology theory which is related to K-theory in roughly the same way as homology is related to cohomology. A Bott periodicity map, Per: $\mathrm{Ext}(S^2X) \to \mathrm{Ext}(X)$, was defined and was proved to be injective for all X and surjective for smooth X. Also $\mathrm{Ext}(X)$ was given the structure of a not necessarily Hausdorff topological group, and the closure of the identity was called $\mathrm{PExt}(X)$.

THEOREM 1. Per *is surjective for all X.*

THEOREM 2. *There is a natural short exact sequence,*

$$0 \to \mathrm{Ext}^1_{\mathbf{Z}}(K^0(X), \mathbf{Z}) \to \mathrm{Ext}(X) \xrightarrow{\gamma_\infty} \mathrm{Hom}(\widetilde{K}^1(X), \mathbf{Z}) \to 0,$$

which splits noncanonically.

COROLLARY. $\mathrm{PExt}(X)$ *is the maximum divisible subgroup of* $\mathrm{Ext}(X)$.

THEOREM 3. *If $\tau_t\colon C(X) \to A$, $0 \leq t \leq 1$, is a continuous family in the sense that $\tau_t(f)$ is continuous for each $f \in C(X)$, then each τ_t defines the same element of* $\mathrm{Ext}(X)$.

For a more leisurely account of these results, see [3]. See also [4], [5], [8]. Ext_* satisfies parallel axioms to the Steenrod homology theory [11], whose axiomatic description in [10] plays a key role in the proofs. Algebraic K-theory

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(cf. [9]) also plays a key role by yielding a natural definition of an isomorphism κ: $\ker \gamma_\infty \longrightarrow \mathrm{Ext}^1_{\mathbf{Z}}(K^0(X), \mathbf{Z})$. κ is defined by applying the algebraic K-theory long exact sequence to $0 \longrightarrow K \longrightarrow E \longrightarrow C(X) \longrightarrow 0$ and obtaining (in part)

$$0 \longrightarrow \mathbf{Z} \cong K_0(K) \longrightarrow K_0(E) \longrightarrow K_0(C(X)) \cong K^0(X) \longrightarrow 0.$$

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In another context [2] we have defined an almost polonais group as the quotient of a polonais (complete, separable, metrizable) group by a normal subgroup which is a continuous homomorphic image of a polonais group. These are not necessarily Hausdorff topological groups with some additional structure, and the abelian ones form an abelian category. Theorem 2 shows that Ext(X) is the direct sum of two almost polonais groups, and we would like to know whether Ext(X) is naturally such an object.

3. **The trace and determinant invariants.** Let $\mathfrak{A}$ be a *-sublagebra of $L(H)$ such that $\mathfrak{A}$ contains the trace class, J, and is commutative modulo J. As in [7], we obtain a symbol map ϕ: $\mathfrak{A} \longrightarrow C(X)$. Here we assume $X \subset \mathbf{R}^n$ and range $\phi = C^\infty(X)$, the algebra of restrictions to X of C^∞ functions on $\mathbf{R}^n$. Let $\widetilde{X}$ be a closed ball containing X. Helton and Howe [7] defined a *trace invariant* l: $\Omega \longrightarrow \mathbf{C}$, where Ω is the space of exact C^∞ 2-forms on $\widetilde{X}$ and $l(df \wedge dg) = \mathrm{tr}(AB - BA)$, where A and B are elements of $\mathfrak{A}$ such that $\phi(A) = f|X$ and $\phi(B) = g|X$. If A and B are invertible, Helton and Howe also considered $\det(ABA^{-1}B^{-1}) = \delta(\phi(A), \phi(B))$. δ is a bimultiplicative form on a subgroup of the group of units in $C^\infty(X)$. In [1] we showed, in the special case $X \subset \mathbf{R}^2$, that δ can be extended to a form d on the whole group of units and that d can be calculated from the trace invariant. As suggested to us by H. Sah, the algebraic properties of d provided an analogy with algebraic K-theory. We will now define a new determinant invariant, d_1: $K_2(C^\infty(X)) \longrightarrow \mathbf{C}^*$, such that d is the restriction of d_1 to the Steinberg symbols.

Consider the short exact sequence, $0 \longrightarrow J \longrightarrow \mathfrak{A} \longrightarrow \mathfrak{A}/J \longrightarrow 0$, and the corresponding algebraic K-theory long exact sequence $\cdots \longrightarrow K_2(\mathfrak{A}/J) \longrightarrow K_1(J) \longrightarrow K_1(\mathfrak{A}) \cdots$. Using the definition of $K_1(J)$ and the most basic properties of the determinant (on the determinant class, $I + J \subset L(H)$), we obtain a map det: $K_1(J) \longrightarrow \mathbf{C}^*$. This pulls back to d': $K_2(\mathfrak{A}/J) \longrightarrow \mathbf{C}^*$. Using analytic techniques (mainly suggested by [7]), we can modify d' to obtain d_1.

The restriction of d_1 to K_2', the range of $K_2(C^\infty(\widetilde{X})) \longrightarrow K_2(C^\infty(X))$ (which is the same as the kernel of $K_2(C^\infty(X)) \longrightarrow K^2(X)$), can be calculated from the trace invariant: Roughly one shrinks $\widetilde{X}$ to a point and differentiates with respect to "time". In this way we obtain a map θ: $K_2(C^\infty(\widetilde{X})) \longrightarrow \Omega$, and $d_1(C) = \exp(l(\theta(\widetilde{C})))$, for $\widetilde{C} \in K_2(C^\infty(\widetilde{X}))$ and C its image in $K_2(C^\infty(X))$. The above leads to an explicit formula for θ. See [6] for the relation and application of this formula to algebraic K-theory. Although $\widetilde{C}$ is not uniquely determined by

C, the restriction of $\theta(\widetilde{C})$ to X is unique. If l vanishes at 2-forms which vanish on X, then we obtain l': $K_2' \longrightarrow \mathbf{C}$. According to [7], this occurs precisely when $\overline{\mathfrak{A}}$ is an element of ker $\gamma_\infty \subset \operatorname{Ext}(X)$, and one can then ask whether l' can be extended to l'': $K_2(C^\infty(X)) \longrightarrow \mathbf{C}$ such that $d_1 = \exp(l'')$. This leads to an element of $\operatorname{Ext}^1_{\mathbf{Z}}(K^0(X), \mathbf{Z})$, which vanishes precisely when l'' exists.

REMARKS 1. Although the construction just completed motivated κ, we do not know whether the two constructions actually agree.

2. The algebra $\mathfrak{A}$ is what Helton and Howe call a "one dimensional" algebra. It would be nice to extend the above to the "k-dimensional" algebras of [7]. There seem to be two difficulties: (a) So far as we know, no existing treatment of K_n for $n > 2$ lends itself to explicit formulas as well as [9]. (b) In the k-dimensional case the determinant invariant ought to be defined on K_{2k}; but if we do what is natural in the context of [7], we get something on K_{k+2} (for $k > 1$). Thus perhaps something is wrong for $k > 2$.

We hope that these difficulties will eventually be surmounted and that the result will be significant mutual enrichment of operator theory and algebraic K-theory.

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