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AN L^1-SPACE FOR BOOLEAN ALGEBRAS AND SEMIREFLEXIVITY OF $L^\infty(X, \Sigma, \mu)$

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This note indicates how one can use the ideas of strict topologies on spaces of continuous functions to, at a single stroke, obtain an extended construction of L^1-spaces without reference to measure, obtain ordinary $L^1(X, \Sigma, \mu)$-spaces as the natural dual of $L^\infty(X, \Sigma, \mu)$ and obtain a view of the dual pairing (L^∞, L^1) that is very much like that of (C, M), where C is a space of bounded continuous functions and M a space of bounded Baire or Borel measures.

Earlier results, [1] and [4], suggest this development. In [1], Buck shows that M, the compact regular Borel measures on locally compact X, results as $(C(X), \beta)'$, where β is the topology on $C(X)$ defined by the seminorms $\|f\|_\xi = \sup\{|f(x)\xi(x)|: x \in X\}$, with $\xi \in C$ vanishing at ∞. In [4], this writer showed how β-methods extend to completely regular X, with $\xi \equiv 0$ over compact sets, or zero sets, in $\beta X \backslash X$. For $X = \{1, 2, \ldots\}$, $l^\infty = C$ and $l^1 = M$, and by [1], $(l^\infty, \beta)' = l^1$. By choosing $\xi \equiv 0$ over certain closed nowhere dense subsets of the appropriate Stone space, we show herein that this result is more than the small coincidence formally expected.

2. **The space $L^1(A)$.** Let A be a Boolean algebra [6] and let S be its Stone space with $\eta(a) \subset S$ denoting the compact-open set corresponding to $a \in A$.

We define an indicator function on A, $\chi: A \longrightarrow C(S)$, by $\chi(a) = \chi_{\eta(a)}$ and let $L^\infty(A)$ be the closed linear span of $\chi(A)$ in $C(S)$ in the $\| \ \|$ (= uniform convergence on S) topology on $C(S)$. In fact, $L^\infty(A) = C(S)$.

For each increasing sequence $a_n \in A$ with $a = \sup a_n$ (i.e., $a_n \nearrow a$), let $Q = \eta(a) \backslash \bigcup_{n=1}^\infty \eta(a_n)$ and define the β_Q topology on $L^\infty = L^\infty(A)$ by the seminorms $\|f\|_\xi$ for $f \in L^\infty$ where $\xi \in C(S)$ and $\xi \equiv 0$ on Q. Let β be the inductive limit topology over all such Q. We remark that β may be neither Hausdorff, nor finer than pointwise convergence and is the $\| \ \|$-topology iff all increasing $\{a_n\}$ with a supremum in A are finite.

We now define $L^1(A)$ by $L^1(A) = (L^\infty(A), \beta)'$, the β-dual of $L^\infty(A)$. It is possible that $L^1(A) = \{0\}$ ([6, p. 65] and (2) below).

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The crucial result is

THEOREM 1. $\chi: A \to (L^\infty(A), \beta)$ *is a vector measure.*

PROOF. If $a_n \rightharpoonup a$ and W is a β-neighborhood of 0, choose $\xi \equiv 0$ on Q so that $\|f\|_\xi \leq \epsilon$ puts $f \in W$. If n_0 is chosen so that $S \setminus \eta(a) \cup \eta(a_{n_0}) \supset \{x: |\xi(x)| \geq \epsilon/2\}$, then $\|\chi(a_n) - \chi(a)\|_\xi \leq \epsilon$ for $n \geq n_0$ and we are done.

Consequently,

THEOREM 2. *The equality* $\mu(a) = (\hat{\mu} \circ \chi)(a)$ *defines a* 1-1, *onto correspondence between the positive elements* $\hat{\mu} \in L^1(A)$ *and the finite-valued positive measures on* A.

PROOF. $\hat{\mu} \circ \chi$ is obviously a measure on A. Conversely, if μ is given, $\phi(\Sigma_{i=1}^n \alpha_i \chi(a_i)) = \Sigma \alpha_i \mu(a_i)$ extends uniquely to a bounded functional on L^∞ which is then β-continuous because μ is a measure.

3. **Spaces** $L^\infty(X, \Sigma, \bar{\mu})$. Let A be the Boolean algebra $\Sigma/\bar{\mu}^{-1}(0)$ under $\cap, \Delta$, where $\bar{\mu}$ is σ-finite. Let [ ] denote equivalence classes in $L^\infty(\bar{\mu})$ or A alternatively. Define (e.g. [6, p. 155]) $\theta: L^\infty(X, \Sigma, \bar{\mu}) \to L^\infty(A)$ by $\theta[\chi_E] = \chi[E]$, extended linearly and by uniform closure to all of $L^\infty(X)$; θ is an $\| \ \|_\infty - \| \ \|$ isometry onto. Let β_∞ be the weak topology on $L^\infty(\bar{\mu})$ making θ β continuous into $L^\infty(A)$; θ is a $\beta_\infty - \beta$ homeomorphism.

THEOREM 3. $(L^\infty(\bar{\mu}), \beta_\infty)' = L^1(X, \Sigma, \bar{\mu}) = \theta'(L^1(A))$.

The proof depends on the fact that $\bar{\nu} \in L'(\bar{\mu})$ allows $\nu[E] = \bar{\nu}(E)$ to be a well-defined measure on A.

THEOREM 4. β_∞ *is the finest locally convex topology on* $L^\infty(\bar{\mu})$ *with dual* $L^1(\bar{\mu})$, *and, moreover, on* $\| \ \|_\infty$*-bounded sets,* β_∞ *agrees with, and is the finest locally convex topology on* $L^\infty(\bar{\mu})$ *to so agree with, convergence in* $\bar{\mu}$*-measure. Consequently, the* β_∞*-continuity of linear maps is determined on such sets by continuity under convergence in* $\bar{\mu}$*-measure.*

The proof of this result is most easily obtained through Theorem 6 below.

4. **Further results.** Among other results we select two which seem to most justify our constructions above. For $\hat{\nu} \in L^1(A)$ let $\|\hat{\nu}\| = \sup\{|\hat{\nu}(f)|: f \in L^\infty, \|f\| \leq 1\}$. For $f \in L^\infty$, let $\phi_f \in (L^1(A), \| \ \|)'$ by $\phi_f(\hat{\nu}) = \hat{\nu}(f)$. By $(L^1(A), \| \ \|)' = L^\infty(A)$ we mean that the map $f \to \phi_f$ is an isometry onto $(L^1(A), \| \ \|)'$. This map is 1-1 iff β is T_2.

THEOREM 5. *If* A *is* σ*-complete and satisfies the countable chain condition and* β *is* T_2, *then* $(L^1(A), \| \ \|)' = L^\infty(A)$.

The proof depends on the fact that $L^1(A)$ coincides with Dixmier's normal measures on S under these hypotheses. The converse may not generally hold; for example if $A = 2^{[0,1]}$ and we assume that no subset of [0, 1] has cardinal of measure zero. Indeed for algebras without the countable chain condition, one should go to the topology $\bar{\beta}$ defined by sets $Q = \eta(\sup B)\backslash \bigcup_B \eta(b)$ where $B \subset A$, whereupon this theory begins to meet that of [3].

THEOREM 6 (DUNFORD-PETTIS). *For a bounded subset H of $L^1(A)$, these are equivalent*:

(1) *H is weak* (i.e., $\sigma(L^1, L^\infty)$) countably compact,*

(2) *$H \circ \chi$ is uniformly absolutely continuous with respect to μ.*

(3) *If $a_n \nearrow a$, then $\|\hat{\nu}_{a_n} - \hat{\nu}_a\| \longrightarrow 0$ uniformly over $\hat{\nu} \in H$, where $\hat{\nu}_b(f) = \hat{\nu}_b(\chi(b)f)$.*

Note that (2) *applied to $H = \{\hat{\nu}\}$, $\hat{\nu}$ fixed in $L^1(A)$ yields*: $\epsilon > 0 \Longrightarrow \exists\, \delta > 0 \ni \mu(a) < \delta \Longrightarrow \nu(a) < \epsilon$.

There are a number of obvious questions remaining, but in particular: When and only when is $L^1(A)$ the L^1-space of a Boolean measure algebra?

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