BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 81, Number 6, November 1975

SOME INVARIANTS OF GENERIC IMMERSIONS AND THEIR GEOMETRIC APPLICATIONS

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Communicated June 20, 1975

1. This is an announcement of some theorems to appear in full detail elsewhere [2].

Let X, Y be smooth manifolds, with $\dim X < \dim Y$ and $f\colon X \longrightarrow Y$ a *generic* immersion. To f we will attach two numerical invariants $\mu_2(f)$ and $\nu_3(f)$, which will be described in the next paragraphs; it is conceivable that a more cohomological, characteristic-classes-type approach to these invariants should be possible. Anyway, granted their definition one has the following results:

THEOREM. *Let $f\colon X \longrightarrow Y$ be a generic immersion as above. The necessary and sufficient condition for the existence of a smooth embedding $X \xrightarrow{F} Y \times R$ lifting f is that $\mu_2(f) = \nu_3(f) = 0$.* $\square$

F "lifts f" means that the following diagram is commutative:

$$\begin{array}{ccc} & & Y \times R \\ & \overset{F}{\nearrow} & \downarrow \\ X & \xrightarrow[f]{} & Y \end{array}$$

COROLLARY 1. *Suppose that $\pi_1 X = 0$. The necessary and sufficient condition for the existence of a smooth embedding $X \xrightarrow{G} Y \times S_1$ lifting f is that $\mu_2(f) = \nu_3(f) = 0$.* $\square$

The next corollary has some connection with the group Θ_3 of Milnor and Kervaire [1]. We consider a smooth homotopy 3-sphere Σ_3 and two points p_0, $p_1 \in \Sigma_3$ $(p_0 \neq p_1)$. We consider two small 2-spheres, in Σ_3, of centers p_0 and p_1: S_2^0, S_2^1. By the Smale-Hirsch immersion theory there is a (generic) regular homotopy:

$$f \in \mathrm{Imm}_1 (S_2 \times I, (\Sigma_3 - \{p_0, p_1\}) \times I)$$

connecting S_2^0, S_2^1. (The subscript I means that f is level-preserving.)

COROLLARY 2. *Let Σ_3 be a smooth homotopy 3-sphere, and f some gener-*

AMS (MOS) subject classifications (1970). Primary 22–XX.

ic regular homotopy as above. If $\mu_2(f) = \nu_3(f) = 0$, *then* Σ_3 *is h-cobordant to* S_3. □

2. We shall define now the invariant μ_2. Let $M^i(f)$ denote the i-tuple points of f at the source X:

$$X \supset M^2(f) \supset M^3(f) \supset \cdots \supset M^i(f) \supset M^{i+1}(f) \supset \cdots$$

$M^i(f)$ is a smooth manifold with singularities. $M^{i+1}(f) \subset M^i(f)$ is exactly the singular set.

Now, for an arbitrary set E, we consider the ith cartesian power E^i, the ith symmetric power $S^iE \xleftarrow{P_i} E^i$, and the two diagonals: $\{x, \ldots, x\} = \operatorname{diag}_i E \subset \operatorname{Diag}_i E \subset E^i$. $f\colon X \longrightarrow Y$ induces $f^i\colon X^i \longrightarrow Y^i$ and we define the i-tuple points at the X^i, or S^iX level, by:

$$M_i(f) = (f^i)^{-1}(\operatorname{diag}_i Y) - \operatorname{Diag}_i X \subset X^i,$$

$$\hat{M}_i(f) = P_i M_i(f) \subset S^iX.$$

Note that $P_i : M_i(f) \longrightarrow \hat{M}_i(f)$ is a covering space, and that the spaces involved are smooth (nonsingular) manifolds. Let $\pi_0 \hat{M}_i(f)$ denote the set of connected components of $\hat{M}_i(f)$. The invariant μ_2 is a function: $\mu_2\colon \pi_0\hat{M}_2(f) \longrightarrow \{0, 1\}$ where $\mu_2(\alpha) = 0$ iff the covering $P_2^{-1}(\alpha) \longrightarrow \alpha$ is trivial. In view of Corollary 2, the following remark might be useful:

PROPOSITION. *Let M, N be manifolds of dimensions* 2 *and* 3, *M closed,* $\partial N = \varnothing$. *M and N are supposed orientable and*

$$\begin{array}{ccc} M \times I & \xrightarrow{\quad f \quad} & N \times I \\ & \searrow \qquad \swarrow & \\ & I & \end{array}$$

is a generic regular homotopy. Then $\mu_2(f) = 0$ *if and only if* $\hat{M}_2(f)$ *is orientable.* □

3. We will define now invariants $\nu_3, \nu_4, \ldots$ which are the "good" generalization of μ_2. Only ν_3 is need here, but it is perhaps more enlightening to define them all.

We consider the inverse system:

$$\begin{array}{ccc} M_2(f) & \overset{d_\alpha}{\underset{d_\beta}{\longleftarrow}} & M_i(f) \\ & \swarrow P_i & \\ \hat{M}_i(f) & & \end{array}$$

where $d_\alpha, d_\beta \cdots$ are the $i(i-1)$ natural ways of an oriented i-tuple point to degenerate into an oriented double point.

The "limit" of this system lies in $M_2 \times M_i \times \hat{M}_i$, and its projection in $M_2 \times \hat{M}_i$ is denoted by $S_{2,i}$. One has a natural projection, which is a covering map $S_{2,i} \xrightarrow{Q_i} \hat{M}_i$ and if $\hat{M}_i \ni x = \{x_1, \ldots, x_i\}$(the set of distinct points $x_1, \ldots, x_i$) then $Q_i^{-1}(x) = x \times x - \operatorname{diag} x$.

The structural group of the fibration Q_i is reduced; instead of being $S(i(i-1))$ it is $S(i) = \operatorname{Perm}\{x_1, \ldots, x_i\}$.

On $S_{2,i}$ we introduce the *equivalence relation* $\sim$ ($\sim_i$) which is, by definition, the smallest equivalence relation with the followign properties:

(a) $(x', x'') \sim (y', y'') \Rightarrow (x'', x') \sim (y'', y')$.

(b) If $(x', x''), (y', y'') \in S_{2,i} \subset M_2 \times \hat{M}_i \longrightarrow M_2$ have their images in the same connected components of M_2, they are equivalent.

(c) Let $x \in \hat{M}_i$ and consider $g \in S(i)$, *a circular permutation* of length i, acting on the fiber $Q_i^{-1}(x)$. If

$$(x_r, x_j) = y \sim g\,y \sim g^2 y \sim \cdots \sim g^{i-2} y = (x_j, x_k),$$

then $(x_r, x_k) \sim (x_r, x_j)$.

If $i = 3$, property (c) means just that

$$(x_r, x_j) \sim (x_j, x_k) \Rightarrow (x_r, x_j) \sim (x_r, x_k).$$

(Note that in (c) everything is in one fiber; in (a), (b) this is not necessarily so.)

We shall define ν_i: $\pi_0\hat{M}_i \longrightarrow \{0, 1, \ldots\}$ $(i \geqslant 3)$, as follows:

If $x \in \alpha \in \pi_0\hat{M}_i$, we consider the number of *distinct* subsets $E \subset Q_i^{-1}(x)$ such that:

(a) E has i elements.

(b) There exists $y = (x_j, x_k) \in Q_i^{-1}(x)$ and $g \in S(i)$, circular permutation of length i, such that $E = \{y, g, y, \ldots, g^{i-1}y\}$.

(c) $y \sim g\,y \sim \cdots \sim g^{i-1}y$.

[Two sets E, E' obtained one from the other by $(x_j, x_k) \longrightarrow (x_k, x_j)$ will not be regarded as distinct.]

The number of distinct E's is, by definition, $\nu_i(\alpha)$. (This number is independent of $x \in \alpha$.)

FINAL REMARKS. (1) The notations of [2] are slightly different from the notations used here. In [2], M_i becomes M_i and $S_{2,i}$ becomes $\mathsf{S}_{2,i}$. M_i from [2] is the set of i-tuple points in $X \times S^{i-1}X$. Looking at the i-tuple points at the level $X \times S^{i-1}X$ means, exactly, blowing up the singularities of $M^i(f) \subset X$ in order to get a smooth manifold (i.e., what one gets at the $X \times S^{i-1}X$ level is the "resolution of singularities" for $M^i(f)$).

(2) We have assumed here that $\dim X < \dim Y$ (or $\dim X = \dim Y$, $\partial Y = \emptyset$, X compact bounded, and $f\,|\,\partial X$ generic). Otherwise, the preceding theory is to be replaced by the following remark: A (connected) covering map $X \xrightarrow{f} Y$

can be lifted to $Y \times R$ if and only if it is infinitely cyclic.

(3) The invariant μ_2 (for the case dim $X =$ dim $Y = 3, \ldots$) turns out to be deeply connected to the handle-body structure of $\Sigma_3 \times I$ [3].

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