BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 81, Number 6, November 1975

RESEARCH ANNOUNCEMENTS

The purpose of this department is to provide early announcement of outstanding new results, with some indication of proof. Research announcements are limited to 100 typed lines of 65 spaces each. A limited number of research announcements may be communicated by each member of the Council who is also a member of a Society editorial committee. Manuscripts for research announcements should be sent directly to those members of the Publications Committees listed on the inside back cover.

LIE GROUPS WITH COMPLETELY CONTINUOUS REPRESENTATIONS

BY L. PUKANSZKY[1]

Communicated by I. M. Singer, July 19, 1975

I. Let G be a separable locally compact group, and da an element of the right invariant Haar measure on G. We say that G is a CCR group if for any continuous, irreducible unitary representation T and for any complex-valued integrable function φ the operator $\int_G \varphi(a)T(a)\,da$ is completely continuous.

One of the principal results of the present note provides a characterization of all connected and simply connected CCR Lie groups (cf. Theorem 3). This extends previous results by Harish-Chandra and Auslander, Kostant and Moore obtained respectively for the semisimple and solvable case. Observe that §§II and III below are independent of each other. All Hilbert spaces occurring in our discussion will be assumed to be separable.

II. Let M be a semifinite factor and Φ a faithful, normal and semifinite trace on M (for references on this and the notions employed below cf., e.g., [3, p. 81ff.]) A positive operator A in M will be called completely continuous if, given its spectral representation $A = \int_0^{+\infty} \lambda\, dE_\lambda$, we have $\Phi(I - E_\lambda) < +\infty$ for all $\lambda > 0$. We say that A is completely continuous if and only if so is $|A|$. We write $C(M)$ for the collection of all completely continuous operators. Let G be a separable locally compact group and $\mathfrak{G}$ its group C^* algebra. We recall (cf. loc. cit.) that a factor representation T, generating M, is called normal if $T(\mathfrak{G}) \cap C(M)$ contains a nonzero operator. We shall say that G *is a* GCCR *group if for all of its normal representations we have* $T(\mathfrak{G}) \subset C(M)$.

AMS (MOS) subject classifications (1970). Primary 22D25; Secondary 46L05, 46L25.

[1]This research was supported by a grant from the National Science Foundation.

Assume now that G is a connected and simply connected Lie group with the Lie algebra $\mathfrak{g}$. We recall that $\mathfrak{g}$ contains a well-determined maximal semisimple ideal $\mathfrak{g}_1$, and that $\mathfrak{g}$ is a direct product of $\mathfrak{g}_1$ and of its centralizer $\mathfrak{g}_2$ in $\mathfrak{g}$. If G_k is the connected analytic subgroup determined by $\mathfrak{g}_k$ $(k = 1, 2)$ of G, we have $G = G_1 \times G_2$. We shall say that G has no semisimple factors (or that G is n.s.f.) if $\mathfrak{g}_1 = 0$, and shall refer to G_2 as the n.s.f. component of G. We recall finally that if $\mathfrak{g}$ is solvable, by definition, its roots are the linear forms associated with the (necessarily one-dimensional) irreducible quotient modules of its adjoint representation acting on $\mathfrak{g}_C$. This being so we have

THEOREM 1. *Suppose that G is a connected and simply connected n.s.f. Lie group with the Lie algebra $\mathfrak{g}$. Then the following four properties are equivalent*: (i) *Any closed prime ideal of the group C^* algebra is maximal.* (ii) *G is a GCCR group.* (iii) *For any $g \in \mathfrak{g}'$ (= dual of the underlying space of $\mathfrak{g}$ under the action of the coadjoint representation) the closure of Gg is composed of orbits of the same dimension.* (iv) *Denoting by $\mathfrak{r}$ the radical of $\mathfrak{g}$, $\mathfrak{g}/\mathfrak{r}$ is compact and all roots of $\mathfrak{r}$ are purely imaginary.*

The equivalence of (i) and (iv) is implied by a recent result, to be published, of C. Moore and J. Rosenberg.

III. The purpose of what we say next is the definition of the reduced stabilizer. Let G be any connected and simply connected Lie group with the Lie algebra $\mathfrak{g}$, $\mathfrak{n}$ the greatest nilpotent ideal of $\mathfrak{g}$, and N the corresponding connected subgroup of G. For some element f of $\mathfrak{n}'$, we denote by π the irreducible representation of N belonging to the Kirillov orbit $Nf \subset \mathfrak{n}'$ (cf. for all this, e.g. [2, Chapitre II, p. 93]); then we have $G_\pi = G_f \cdot N$. Let α be an N invariant extension cocycle of π to G_π. As M. Duflo has shown (cf. loc. cit. p. 109), there is a canonically constructed covering $\widetilde{G}_f$, of order not exceeding two, of G_f, such that $\alpha|\widetilde{G}_f \times \widetilde{G}_f$ is a coboundary. This being said, for some fixed $g \in \mathfrak{g}'$ let us put $f = g|\mathfrak{n}$ and $\mathcal{G} = \widetilde{G}_f$; the latter, through its projection onto G_f, acts on $\mathfrak{g}'$ and thus we can form $\mathcal{G}_g$. The connected component of the identity $(\mathcal{G}_g)_0$ is a covering of order $\leqslant 2$ of $(G_g)_0$. We shall say that *g is admissible if there is a character χ_g of $(\mathcal{G}_g)_0$ such that $d\chi_g = i(g|\mathfrak{g}_g)$ and, when $(\mathcal{G}_g)_0$ is a double covering of $(G_g)_0$, there is an $\epsilon \in (\mathcal{G}_g)_0$ over the unity such that $\chi_g(\epsilon) = -1$.* We denote by $\mathcal{W}$ $(\subset \mathfrak{g}')$ the totality of all admissible elements; $\mathcal{W}$ is evidently G invariant. If $g \in \mathcal{W}$, $\ker(\chi_g)$ is invariant in $\mathcal{G}_g$; we denote the complete inverse image of the center of $\mathcal{G}_g/\ker(\chi_g)$ by $\overline{\mathcal{G}}_g$, and write $\overline{G}_g$ for the direct image of the latter in G_g. We shall call $\overline{G}_g$ the *reduced stabilizer* of g $(\in W)$.

THEOREM 2. *Assume that G is a connected and simply connected Lie group with the Lie algebra $\mathfrak{g}$ such that the radical is cocompact. Then G is of type* I *if and only if $\mathcal{W}/G$ is a T_0 space and $\overline{G}_g$ is cofinite in G_g.*

This result extends a theorem of Auslander and Kostant (cf. [1, Theorem V.3.2, p. 351]).

IV. THEOREM 3. (a) *A connected and simply connected Lie group G is CCR if and only if its n.s.f. component* (*cf.* §II) *is so.* (b) *Suppose that G has no semisimple factors. Then it is CCR if and only if* $\mathcal{W}/G$ *is a* T_1 *space, and* $\overline{G}_g$ *is cofinite in* G_g *for each* $g \in \mathcal{W}$.

BIBLIOGRAPHY

1. L. Auslander and B. Kostant, *Polarization and unitary representations of solvable Lie groups,* Invent. Math. 14 (1971), 255–354. MR 45 #2092.

2. M. Duflo, *Sur les extensions des représentations irréductibles des groupes de Lie nilpotents,* Ann. Sci. École Norm. Sup. (4) 5 (1972), 71–120. MR 46 #1966.

3. L. Pukanszky, *Characters of connected Lie groups,* Acta Math. 133 (1974), 81–137.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF PENNSYLVANIA, PHILADELPHIA, PENNSYLVANIA 19174