In the last two chapters weak (distributional) solutions of elliptic systems in two independent variables are discussed. The treatment of these establishes contact with earlier work by Bers (pseudo-analytic functions) and Douglis (hyperanalytic functions).

The investigations presented in this volume are highly technical and complex. It is probably inevitable, but from the reader's point of view far from welcome, that the presentation involves an unfavourable ratio of (often lengthy and involved) formulae to conceptual exposition. The series in which this volume appeared aims at rapid publication of topical material and is prepared to tolerate imperfection, but accepting this, one yet wishes that typing and proofreading had been more careful.

Notwithstanding these imperfections, this volume will be of considerable value to those engaged in research in related areas.

A. ERDÉLYI

The structure of fields, by David J. Winter, Graduate Texts in Mathematics, 16, Springer-Verlag, New York, 1974, xii+205 pp., $12.80.

David J. Winter's book, *The structure of fields*, is written in the form of a graduate level text book. The preface gives a glowing statement of objectives:

> This book is written with the objective of exposing the reader to a thorough treatment of the classical theory of fields and classical Galois theory, to more modern approaches to the theory of fields and to one approach to a current problem in the theory of fields, the problem of determining the structure of radical field extensions.

This statement is very misleading. It is true that there are chapters on basic algebra and group actions, elementary field theory, the structure of algebraic extensions, classical Galois theory, algebraic function fields and Galois theories involving algebraic structures other than groups, and a series of appendices supplementing these, but the presentation seems completely unsuitable for a beginning graduate student, or even for a more advanced student's first contact with field theory. Winter indicates that much of the material is based on courses he has taught on bialgebras and field theory. For others reaching either of these courses, Winter's book might be used as a supplementary reference, of value particularly for its exercises and its discussion of the bialgebra Galois theory of purely inseparable extensions of a field. However, I would not recommend it as the main text. For an advanced course on field theory, Jacobson, *Lectures in abstract algebra*. Vol. 3: *Theory of fields and Galois theory*, Van Nostrand, Princeton, N.J., 1964, covers practically all of the material in five out of the six chapters of Winter's book, and also includes several important topics not mentioned by Winter. Not only is his coverage more thorough, but Jacobson also makes significantly more of an effort to consider the student's viewpoint than

Winter does. For an advanced course on bialgebras, considerably more work should be included than the appropriate two appendices in Winter. For example, Sweedler, *Hopf algebras*, Benjamin, New York, 1969, is a better candidate for a basic text. Winter has written an extremely exciting preface, but the book simply does not live up to it.

In this review of Winter's book, I will first indicate the drawbacks which would probably discourage first year graduate students and those who compulsively read all books from the beginning, and then discuss how the book might be used as a reference for portions of an advanced course.

The opening thesis in the preface is:

> The theory of fields is one of the oldest and most beautiful subjects in algebra. It is a natural starting point for those interested in learning algebra, since the algebra needed for the theory of fields arises naturally in the theory's development and a wide selection of important algebraic methods are used. At the same time, the theory of fields is an area in which intensive work on basic questions is still being done.

While one may quarrel with some points in this, such as field theory being a natural starting place for the study of algebra, there is much truth in the paragraph. Historically one of the well-springs of the development of algebra was the problem of finding roots of polynomials, and field theory was developed to attack this problem. Abstract field theory yielded proofs that no ruler and compass construction can square the circle or trisect a 60° angle, and that there is no formula for the roots of a polynomial of degree ≥ 5 expressible in terms of the operations of addition, subtraction, multiplication, division, and the extraction of nth roots. The complex numbers, now a very applicable as well as theoretical mathematical concept, were originally studied because of a desire to be able to solve the equation $X^2 + 1 = 0$ in a "nice number system" containing the real numbers. For "nice number system" read "field". Unfortunately, the text of this book reflects very little of this. The reader will find that motivation and concrete evidence that field theory is more than a collection of definitions, theorems, and proofs is almost exclusively relegated to the exercises at the ends of the chapters. A book on "one of the oldest and most beautiful subjects in algebra" should give the reader some idea of why the subject is old and beautiful, how it arose, what it is good for, and what techniques and ideas are most significant. Examples are essential to motivate and illustrate theorems, to indicate that some questions that arise naturally from the theory have negative answers, to show that the theorems are not discussing the empty set of objects, to show why certain hypotheses are essential in theorems, and to justify making definitions of all kinds. Very few examples are given in the text, many more in the exercises, and even more must be provided by the instructor or student.

The objective of thorough coverage of the classical theory of fields is not met by this book. While studying commutative field extensions of base fields is one extremely important aspect of field theory, it is by no means the only one. Any thorough treatment of field theory should consider such topics as

valuated fields, real closed fields, the Brauer group, crossed products, and should seriously develop enough number theory to serve as motivation for and applications of field theoretic results. Winter does not. *Algebra,* by Serge Lang, Addison Wesley, Reading, Mass., 1965, contains broader coverage of classical field theory than Winter, and it is a general graduate algebra text, not a field theory text. Jacobson, *Lectures in abstract algebra.* Vol. 3: *Theory of fields and Galois theory,* Van Nostrand, Princeton, N.J., 1964, covers even more than Lang. It is true that these are larger books than Winter's, and trying to condense a large amount of material into a small space necessitates omissions, but that means the treatment is a very long way from being thorough. The student would get more classical field theory by using one of the above texts. A more accurate description of this book's actual coverage is Galois theory and related field theory. It might also indicate that the text is almost completely abstract theory. With a few exceptions, such as finite fields and cyclotomic extensions of the rationals, the body of the text does not deal with the structure of specific fields at all.

Winter's style is Definition-Theorem-Proof with an extremely small amount of exposition. For example, the first five sections of Chapter 1, p. 26 to the middle of p. 36, have 43 items given numbers and labeled **definition, proposition, theorem,** or **corollary** and all of 97 lines not in one of the above or a labeled proof. A large portion of these 97 lines are unlabeled definitions or proofs, and there are only 30 or so that I would classify as exposition (I exclude pre- or restatements of propositions from exposition in this count unless they shed additional light on the meaning of the proposition). The first three sections of Chapter 3, p. 65 to the middle of p. 73, have only 14 numbered items and considerably more expository material. They happen to be a nice section of the book. That is not maintained. The first three sections of Chapter 6 have counts much closer to Chapter 1 than Chapter 3. Moreover, if one concludes that separating items out as numbered definitions or propositions is an indication of importance, then one would have to conclude that the Frobenius homorphism and the Hilbert Nullstellensatz, which only rate a few unnumbered lines and reference to one exercise apiece, are rather unimportant items. In contrast, many items are set off as numbered propositions that have one line proofs.

When a complicated or unintuitive proof is substituted for some more straightforward proof, a reader's difficulties in studying a math text are increased. Winter does more than his share of this. Two examples are worth citing for different reasons.

In his last section of the chapter on classical Galois theory, Winter finally gets around to a discussion of the existence of polynomials with rational coefficients which are not solvable by radicals. He selects a straightforward program to obtain such polynomials. Pick your favorite prime q. To assure nonsolvability, q should be $\geqq 5$, but Winter expects the reader to fill in that detail. Pick your favorite complex conjugate pair r_1 and r_2 such that their sum and product are integers. Pick your favorite $q-2$ distinct integers $r_3, \cdots, r_q$. Then form the polynomial of degree q with the r_i as roots. So far the approach is excellent for a classroom situation. While reading it, the

student can supply his own numbers, and end up with a specific polynomial such as x^5-x. Winter's next step is to modify the constant term by adding $1/q_0$ for some prime q_0 such that the resulting polynomial is irreducible by Eisenstein's criterion and still has precisely $q-2$ real roots. Then elementary theory shows the Galois group is the symmetric group on q letters which is not solvable for $q \geqq 5$. This is the kind of thing that most graduate students I know would really appreciate. What then is the problem? How do you find q_0? All Winter says is that, since the roots of a complex polynomial are continuous functions of the constant term, q_0 must exist. What was rather concrete has its punch line completely abstract. Why not draw a picture, as Jacobson does in his text mentioned above, and say that $1/q_0$ must be less than the smallest distance from the x-axis to a local maximum or minimum. Freshman calculus can then be used to calculate q_0 without appeal to an abstract existence argument. In the above example, one gets that $x^5-x-\frac{1}{3}$ has Galois group S_5. Winter does not include a specific polynomial with rational coefficients to which his construction applies, even in the exercises. His one irreducible quintic, $x^5+5x-10$ in Exercise 3.36, is monotone increasing (calculus again) and so has only one real root.

A second example of a complicated proof replacing a better standard proof is in Winter's construction of a splitting field for a set of nonconstant polynomials. The usual proof is a Zorn's lemma argument restricting all fields looked at to be elements of a specific set for foundational reasons. Winter replaces the Zorn's lemma argument by a much more involved one using transfinite induction and direct limits. Furthermore, he drops the restriction on all fields under discussion being elements of a given set. That makes the argument invalid in most standard set theories. Functions can be defined by induction only from other functions. The other function indicated here is a choice function on a nonempty family of nonempty proper classes. Even in Gödel-Bernays set theory where one can talk about proper classes, one cannot have families of them, so one needs an axiom such as the universe can be well ordered ($V=L$ implies this) to apply Winter's complicated argument. That is an unnecessarily strong axiom.

The last drawbacks to be mentioned are the typical problems of misprints or other annoying minor errors in crucial places including some exercises, excessive terseness in some proofs, belaboring of the obvious in many instances, too involved notation on occasion, and relying on exercises for proofs in the book when the text does not particularly prepare the student to do those exercises.

Now for suggested ways to use the book constructively. If the reader does not have a solid background in graduate algebra and field theory, or if he insists on starting in Chapter 0 or Chapter 1, he or she is likely to get discouraged very early and not read the book. Hence I am assuming that background. There is a thorough discussion of cyclic extensions in the presence of enough roots of unity, so the reader who has not seen this material can benefit, provided he is willing to read through several sections which contain definitions of significant concepts which are essentially not

developed. For example, Galois cohomology in Winter's book consists of
definitions of 1-cocycles, 1-coboundaries, the appropriate equivalence rela-
tion giving H^1, and showing H^1 is trivial in some special cases. This provides
the first proof in the book of Hilbert's Theorem 90 and its additive
analogue. When they are to be used in the discussion of cyclic extensions,
proofs without the excess terminology are given. The instructor could
expand on this material to develop these concepts, or he could use Jacobson
as a text. He might then work in Winter's book by having his students do the
following:

Skip Chapters 0 through 2. Do, however, look over the *exercises* of these
chapters and of succeeding chapters as you read portions of them. The
reader will find in these exercises examples to work on, outlines of proofs of
significant results such as the fundamental theorem of algebra, definitions of
resultants and discriminants and other tools of algebraic number theory,
definitions and properties of perfect fields, and many other things he might
have missed in his previous work. The exercises are at all levels, many not
needing hints, others with hints, still others without any clue in the book as
to their solution. If the reader cannot solve an exercise the text probably will
not help. If he feels the exercise is important, other sources will have to be
consulted.

The reader should start reading text with the first three sections of
Chapter 3. Here he will find a proof of the Galois correspondence theorem
by Galois descent, introducing an approach different from the traditional
ones. For algebraic extensions, groups of automorphisms give information
about a field extension. The third section of Chapter 4 looks at separably
generated, finitely generated field extensions, and is a review of a portion of
Jacobson appropriate at this point in the course. In this case, it is a different
algebraic structure, namely Lie algebras of derivations, that gives informa-
tion on the extension. The student can then go to the heart of the book, §5.3
and Chapter 6, covering Galois theory associating bialgebras with a field
extension. Automorphisms give no information on purely inseparable exten-
sions, but an appropriate bialgebra does. This book introduces the terminol-
ogy used in some modern approaches to the problem of studying purely
inseparable extensions. Translating a major result back into more tradi-
tional language, a purely inseparable extension K of a field k splits as a
tensor product of simple extensions if and only if for some bialgebra T of
simultaneously diagonalizable linear transformations of K over k, satisfies
$t(xy)=t(x)y$ for all $t \in T$ and $x \in K$ if and only if $y \in k$. An exercise outlines a
proof of Sweedler's result that if k is the set of constants of a set of higher
derivations of K over k, then K is such a tensor product. Here Winter's text
is filling in recent results and terminology not found in either Jacobson's or
Sweedler's books. Hence, in a highly specialized course concerned with the
machinery of looking at field extensions, the book serves as a source
complete with exercises of some material omitted from other books that
might be used as the main text in the course.

BARBARA L. OSOFSKY