BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 81, Number 5, September 1975

ERGODIC EQUIVALENCE RELATIONS, COHOMOLOGY, AND VON NEUMANN ALGEBRAS

BY JACOB FELDMAN[1] AND CALVIN C. MOORE[2]

Communicated April 3, 1975

1. **Introduction.** Throughout, $(X, \mathcal{B})$ will be a standard Borel space, G some countable group of automorphisms, R_G the equivalence relation $\{(x, g \cdot x), g \in G\}$, and μ a σ-finite measure on X. For μ quasi-invariant, the orbit structure of the action has been studied by Dye [4], [5], Krieger [8]–[13], and others. Here, ignoring G and focusing on R_G via an axiomatization, and studying a cohomology for R_G, we obtain a variety of results about group actions and von Neumann algebras. The major results are stated below.

2. **Equivalence relations.** R will be an equivalence relation on X with all equivalence classes countable, and $R \in \mathcal{B} \times \mathcal{B}$.

THEOREM 1. *Every R is an R_G.*

Properties of G-actions translate into properties of R_G which can be stated with no G in sight, e.g., quasi-invariance, ergodicity. Let μ be quasi-invariant, and let $\mathcal{C} = \mathcal{B} \times \mathcal{B}|_R$ and $P_l(x, y) = x$, $P_r(x, y) = y$. Now $\mathcal{C}$ has a natural measure class as follows:

THEOREM 2. *The formula $\nu_l(C) = \int |P_l^{-1}(x) \cap C| d\mu(x)$, where $|\cdot|$ is cardinality, and a similar formula for ν_r define equivalent σ-finite measures on $\mathcal{C}$.*

The Radon-Nikodym derivative is the function $D = d\nu_r/d\nu_l$ on R. If $R = R_G$, then $d(\mu \cdot g)/d\mu(x) = D(x, gx)$. Moreover, D is a cocycle in that $D(x, y)D(y, z) = D(x, z)$ a.e. and the D' arising from a μ' equivalent to μ is cohomologous to D.

For ergodic R, one has a classification into types which are I_n, $n = 1, \ldots, \infty$, II_1, II_∞ and III as in [3]. For $j = 1, 2$, relations R_j on $(X_j, \mathcal{B}_j, \mu_j)$ are isomorphic if there is a Borel isomorphism a: $X_1 \to X_2$ with $\mu \sim \mu \circ a^{-1}$ and $R_2(a(x)) = a(R_1(x))$ a.e. If the R_j are ergodic, they are principal groupoids and, hence, define virtual groups [14].

THEOREM 3. *R_1 and R_2 define isomorphic virtual groups iff each is isomorphic to a restriction of the other, where the restriction of R to H is $R \cap H \times H$. Hence, the two notions of isomorphism coincide if R_1 and R_2 are both of infinite type.*

Hyperfiniteness in terms of R becomes: $\exists\, R_n \uparrow R$ with $|R_n(x)|$ finite $\forall n$, $\forall x$.

AMS (MOS) subject classifications (1970). Primary 22D25, 22D40, 28A65, 46L10.

[1] Supported in part by National Science Foundation Grant MPS-75-05576.
[2] Supported in part by National Science Foundation Grant MPS-74-19876.

3. **Cohomology**. For simplicity assume that R is ergodic and let $R^n = \{(x_0, \ldots, x_n), x_0 \sim \cdots \sim x_n\} \subset X^{n+1}$ with the natural measure class generalizing that of Theorem 2. An R module is an abelian polonais group with a Borel map u of R into $\mathrm{Aut}(A)$ with $u(x, y)u(y, z) = u(x, z)$. Define cochain groups $C^n(R, A)$ as the Borel functions mod null functions from R^n to A with coboundary operators $(\delta_n c)(x_0, \ldots, x_{n+1}) = \Sigma_j(-1)^j c(x_0, \ldots, \hat{x}_j, \ldots, x_n)$ if $u = 1$ and with a slight modification if $u \neq 1$. We define cohomology groups $H^n(G, A)$ of this complex; also for $n = 1$, we allow A to be nonabelian and obtain a cohomology set. These groups were introduced in the virtual group context by Westman [17]. We show how to axiomatize these groups and show that they are unique solutions to a universal problem. If $R \sim R_G$ with G acting freely, one may identify $H^n(R, A)$ with $H^n(G, U(X, A))$, where $U(X, A)$ is Borel functions mod null functions from X to A with G operating suitably. If R is hyperfinite and not type I_n so that $R = R_Z$, with Z acting freely, then $H^n(R, A) = 0$ for all $n \geqslant 2$. Since any action of an abelian group is hyperfinite (Dye [5], Feldman and Lind [6]), one can obtain results of the following kind:

THEOREM 4. *If s and t are commuting ergodic independent ($s^n \neq t^m$) automorphisms of (X, μ), then for any Borel function f from X to the circle T, there exist Borel functions g and h to T so that $f = ((g \circ s)/g)((h \circ t)/h)$ a.e.*

Generalizing Mackey [14], we define for $c \in Z^1(R, A)$ a relation $R(c)$ on $X \times A$ by $(x, a) \sim (x^1, a^1)$ iff $x \sim x^1$ and $c(x, x^1)a = a^1$, where A is an abelian locally compact R module with trivial action. Then A acts by right translations on $X \times A$ and preserves $R(c)$ and so acts via Mackey's point realization theorem on $Z = X \times A/\widetilde{R(c)}$, where $\widetilde{R(c)}$ is a countably separated equivalence relation containing $R(c)$ whose image in the measure algebra of $X \times A$ coincides with the $R(c)$ invariant sets. This ergodic action of A is called the range of c, and depends only on the class of c. The isotropy group A_z of A at $z \in Z$ is an a.e. constant closed subgroup $A(c)$ which is called the proper range of c. Now if A^* is the one point compactification of A, we generalize [12] and define the asymptotic range $r^*_\infty(c)$ as the intersection over all subsets B of X of positive measure of the essential ranges in A^* of c restricted to $B \times B$, and $r^*(c) = r^*_\infty(c) \cap A$. An important result is

THEOREM 5. *$r^*(c)$ is a closed subgroup of A depending only on the class of c and equals the proper range $A(c)$ of c.*

For $A = R$ and $c = \log D$, this was done by enumeration of cases in [7], and there is some overlap with results in [2]. We also have

THEOREM 6. *For $A = \mathbf{R}^n + \mathbf{Z}^m$, $c \sim 0$ iff $\infty \notin r^*_\infty(c)$.*

As a corollary we obtain the result that if $\log D$ is bounded, then there is an equivalent invariant measure, a result that also follows from Theorem 1 and [15].

4. **von Neumann algebras.** Generalizing the Zeller-Meier generalization [18] of the Murray-von Neumann group measure space factors, we construct for an ergodic relation R and $t \in H^2(R, T)$ a factor $M(R, t)$ which we view as the "twisted algebra of matrices over R". For $t = 1$ this factor is constructed (less transparently) in [10]. Our Hilbert space H is $L_2(R, \nu_l)$, and we pick $c \in t$ normalized to be skew symmetric. For $F \in L^\infty(R)$ which is band limited in that $|\{x | F(x, y) \neq 0$ and $0 \neq F(y, x)\}|$ is bounded, one defines an operator M_F on H by

$$(M_F f)(x, z) = \sum_{y \sim x} f(x, y) F(y, z) c(x, y, z).$$

These operators form a *-algebra whose weak closure is a factor $M(R, t)$ depending only on t and not on c. The commutant has a similar form. The indicator function of the diagonal Δ is a separating and cyclic vector, and the diagonal subalgebra $A = \{M_F, F = 0 \text{ off } \Delta\}$ is a maximal abelian subalgebra which is regular by Theorem 1. Moreover, there is a normal faithful conditional expectation E of $M(R, t)$ onto A. If M is any factor with abelian subalgebra A satisfying these conditions, we call A a Cartan subalgebra [19]. One of our major results is a converse of this construction.

THEOREM 7. *If A is a Cartan subalgebra of the factor M, then $M = M(R, t)$ for suitable R and t with A as diagonal subalgebra for any R'.*

Of course, if M is a finite factor, the E always exists. One may ask if $M(R, t)$ determines R and t. If we restrict to hyperfinite R (where $t = 1$ automatically), then $M(R, 1)$ does indeed determine R by [4], [5], [6]. A major open problem is whether we get all factors as $M(R, t)$'s. We note that Connes [1] constructs an $M(R, t)$ which is not an $M(R', 1)$.

Our final results concern automorphisms and conjugacy questions. If A is the diagonal subalgebra of $M = M(R, t)$, let $\mathrm{Out}(M, A)$ be the subgroup of the "outer" automorphism group of M which maps A into something inner conjugate to A. Let $\mathrm{Out}(R, t)$ be the group of "outer" automorphisms of the relation R fixing the cohomology class t. We have a structure theorem for $\mathrm{Out}(M, A)$ generalizing results in [16].

THEOREM 8. *We have an exact sequence* $1 \to H^1(R, T) \to \mathrm{Out}(M, A) \to \mathrm{Out}(R, t) \to 1$.

Finally, let A_i be two Cartan subalgebras of M with conditional expectations E_i. The restriction of E_1 to A_2 gives rise to a unique positive measure γ on $X_1 \times X_2$ (where $A_i = L^\infty(X_i, \mu_i)$) whose disintegration products γ_x $(x \in X_1)$ with respect to projection to X_1 are determined by $E_1(a)(x) = \int a(y)\, d\gamma_x(y)$ a.e. for $a \in A_2 = L^\infty(X_2, \mu_2)$. Let us say that A_2 is discrete over A_1 if a.a. γ_x are atomic measures.

THEOREM 9. *If M is an infinite factor, A_1 and A_2 are inner conjugate iff A_2 is discrete over A_1 and A_1 is discrete over A_2.*

ADDED IN PROOF. Theorem 5, for hyperfinite R, was also obtained by K. Schmidt (*Cohomology and skew products of ergodic transformations*, University of Warwick, Coventry, England, preprint).

REFERENCES

1. A. Connes, *A factor not antiisomorphic to itself*, Queens Mathematical Reprint #1974–26.

2. A. Connes and M. Takesaki, *Flot des poids sur les facteurs de type* III, C. R. Acad. Sci. Paris Sér. A–B 278 (1974), A945–A948.

3. Dang-Ngoc-Nghiem, *On the classification of dynamical systems*, Ann. Inst. Poincare 9 (1973), 397–425.

4. H. A. Dye, *On groups of measure preserving transformations*. I, Amer. J. Math. **81** (1959), 119–159. MR 24 #A1366.

5. ———, *On groups of measure preserving transformations*. II, Amer. J. Math. **85** (1963), 551–576. MR 28 #1275.

6. J. Feldman and D. Lind, *Hyperfiniteness and the Halmos-Rokhlin theorem for abelian group actions*, Proc. Amer. Math. Soc. (to appear).

7. T. Hamachi, Y. Oka and M. Osikawa, *Flows associated with ergodic non-singular transformation groups* (preprint).

8. W. A. Krieger, *On non-singular transformations of a measure space*. I, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **11** (1969), 83–97. MR 39 #1628.

9. ———, *On non-singular transformations of a measure space*. II, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **11** (1969), 98–119. MR 39 #1628.

10. ———, *On constructing non-*-isomorphic hyperfinite factors of type* III, J. Functional Analysis **6** (1970), 97–109. MR 41 #4260.

11. ———, *On a class of hyperfinite factors arising from null-recurrent Markov chains*, J. Functional Analysis **7** (1971), 27–42. MR 43 #938.

12. ———, *On the Araki-Woods asymptotic ratio set and nonsingular transformations*, Lecture Notes in Math., vol. 160, Springer-Verlag, New York, 1970, pp. 158–177.

13. ———, *On ergodic flows and the isomorphism of factors* (preprint).

14. G. W. Mackey, *Ergodic theory, and virtual groups*, Math. Ann. **166** (1966), 187–207. MR 34 #1444.

15. J. M. Rosenblatt, *Equivalent invariant measures*, Israel J. Math. **17** (1974), 261–270.

16. I. M. Singer, *Automorphisms of finite factors*, Amer. J. Math. **77** (1955), 117–133. MR 16, 597.

17. J. J. Westman, *Cohomology for the ergodic actions of countable groups*, Proc. Amer. Math. Soc. **30** (1971), 318–320. MR 43 #6402.

18. G. Zeller-Meier, *Produits croisés d'une C*-algèbre par un groupe d'automorphismes*, J. Math. Pures Appl. (9) **47** (1968), 101–239. MR 39 #3329.

19. A. M. Veršik, *Nonmeasurable decompositions, orbit theory, algebras of operators*, Dokl. Akad. Nauk SSSR **199** (1971), 1004–1007 = Soviet Math. Dokl. **12** (1971), 1218–1222. MR 44 #4537.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, BERKELEY, CALIFORNIA 94720