an unfamiliar, but certainly valuable, point of view. The book should be in the hands of anyone working in nilpotent group theory (in the widest sense) and of anyone interested in seeing homological techniques put to work in group theory.

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Aspects of constructibility, by Keith J. Devlin, Lecture Notes in Math., Vol. 354, Springer-Verlag, New York, 1973, xii+240 pp., 22DM-

Gian Carlo Rota [1] named the years 1930–1965 as "The Golden Age of Set Theory." Some of the results presented in this book call for extending the Golden Age beyond 1965. What made 1965 seem like a natural end to the Golden Age is, of course, P. J. Cohen's proof (1963) of the independence of the axiom of choice and of the continuum hypothesis. What followed was a rich crop of independence proofs which showed that a long list of "classical" problems were not solved exactly because they cannot be decided on the basis of the presently accepted axiom systems for set theory. The fact that there are statements in set theory which are independent of the axiom system is no surprise to anybody aware of Gödel's Incompleteness Theorem. The remarkable fact is that the problems which are now proven to be independent are more "natural" in the sense that they are raised by mathematical practice and not especially conceived so as to show independence. We do not claim that the problem of consistency of set theory is not natural for the student of foundations, but that for some reason the working mathematician is not bothered by it.

One direction (the consistency) of the independence of the continuum hypothesis was proved by Gödel in 1938 by introducing the constructible universe, which is a subclass or subcollection of the class of all sets (not necessarily a proper subclass). Gödel showed that the constructible universe is a model of all the axioms of set theory (including the axiom of choice). In fact, this class of the constructible sets (usually denoted by L) is the smallest such model including all the ordinals. Besides, L satisfies the generalized continuum hypothesis, i.e. $2^{\aleph_\alpha}=\aleph_{\alpha+1}$ for every infinite cardinal $\aleph_\alpha$. Gödel also noted that some other problems are settled once we restrict ourselves to the universe of the constructible sets, i.e. we assume $V=L$ where V is the class of all sets.

It will not be an exaggeration to suggest that after the introduction of L by Gödel, the major part of the study of L is due to one person, Ronald B. Jensen. Jensen's results in the 60's and 70's show that the majority of the problems which were shown to be independent of the axioms of set theory are settled once we assume $V=L$. (That includes Souslin Problem, Kurepa Hypothesis, different partition problems, two cardinals problems in model theory, etc.) We do not claim, of course, that every problem in Set Theory is settled by $V=L$, Gödel's Theorem forbids!, but in some ill defined sense, every "natural" problem seems to be decided in the constructible universe.

We do not wish to imply that the solution is always in the "nice" direction. In fact, for many cases the solution of the problem is achieved by constructing in L a simple counterexample to the expected theorem. Thus there is in L a PCA set which is not Lebesgue measurable, the existence of which is otherwise unprovable in set theory (PCA=projection of a complement of analytic set), in L there is a counterexample to Souslin hypothesis, etc. In spite of this last remark, which can be used together with much better arguments against adopting $V=L$ as an extra axiom of Set Theory, we think that it is a remarkable fact that one can get so much more information by assuming $V=L$.

The stated objective of this book is "to collect together most of the existing information about constructibility," but like many of the Springer-Verlag Lecture Notes in Mathematics, "the book is supposed to be essentially a first draft of a proposed book on constructibility theory." There is no doubt that the book achieves its objective. At the present time, there is no other book which contains the recent work on the structure of L besides less recent work which was never grouped in one book. Some of the material was never published before. Also, the standard of exposition is much above what you may expect from a first draft.

The first chapter is a brief survey of the Zermelo-Fraenkel axiom system for set theory. Basic definitions and results are presented and the Lévy hierarchy of formulas of set theory is described. Though officially no set theoretical prerequisites are made it is probably too condensed for a first introduction to the subject. (We do not find this a drawback since we believe that the book is going to be used mainly by more advanced students.)

The author then introduces the constructible hierarchy by which sets are generated step after step in terms of previously constructed sets. It is proved that the class of these sets (L) is a model of all the axioms of set theory including the axiom of choice. The primitive recursive set functions are introduced and probably the most important lemma for the study of L is proved. It is the "condensation lemma" which vaguely states that the construction of L is so uniform, so that at uncountable stages of the construction we had previous stages which are a good approximation to the present stage. More formally, if L_α is the collection of sets constructed up to the α stage and $X \subseteq L_\alpha$, X is an elementary substructure of L, then X is isomorphic to some L_β for $\beta \leqq \alpha$. This lemma (due to Gödel) plays a key role in subsequent chapters and in the present chapter, proving the generalized continuum hypothesis in L.

The next chapter (Chapter 3) presents the solution of the Souslin problem in L. Souslin asked the following problem: Is every complete dense linear ordering with no first or last element such that every family of mutually disjoint intervals is countable, is order isomorphic to the reals? The problem is independent of set theory (Solovay-Tenenbaum, Jech) but in L one can settle the problem by constructing a counterexample. Condensation lemma type of arguments are heavily used in order to destroy at the countable stages any prospective uncountable family of mutually disjoint open intervals in the constructed linear ordering. (Actually we do not construct the linear ordering directly but a so-called Souslin tree, whose existence is equivalent to the existence of a counterexample to the Souslin Hypothesis.)

The way the proof is presented is typical of other Jensen's proofs described in the book: First a certain combinatorial principle (called $\diamond$ in this case) is defined. It is proved that $\diamond$ implies the negation of Souslin's hypothesis and that $V=L$ implies $\diamond$. We find this way, of first extracting the right combinatorial principle from $V=L$ and then getting the theorem from the combinatorial principle, very instructive. Moreover, these combinatorial principles are much more likely to be used by nonlogicians in finding other applications of the same techniques than a direct proof using the definability properties of L.

The next combinatorial problem considered is the existence of a Kurepa tree (a tree with $\aleph_1$ levels; each level is countable but the tree has $\aleph_2$ branches). The existence of such a tree is independent of set theory (Silver, assuming inaccessible cardinal, otherwise it cannot be proved). The proof of Kurepa Hypothesis in L again extracts the right combinatorial property ($\diamond^+$ in this case).

The author goes on to describe some of the implications of assuming $V=L$ on the structure of the reals. In particular Shoenfeld's Absoluteness Theorem is proved as well as the existence of a nonmeasurable PCA set in L.

We now get into the "Fine Structure of L", and the book begins preparing the ground by introducing the Jensen hierarchy (J_α) of the constructible sets which give the same class of sets by slightly different steps of construction. Though not essential, it is more adaptable to the technical arguments which follow and which are the key technical results about the fine structure of the Jensen hierarchy. The most important result is that every J_α is $\sum_n$ uniformiable. This chapter (Chapter 7) is one of the most deep and technically subtle chapters.

The book now applies the $\sum_n$ uniformization theorem to prove in L the combinatorial principle $\square$ which like its predecessors $\diamond$ and $\diamond^+$ has the general character of stating the existence of a uniform sequence of approximation for large sets using small sets.

The next chapters (9–11) generalize Souslin's and Kurepa's Hypotheses to other cardinals ($>\aleph_1$) and show, using $\square$- and $\diamond$-like principles that almost every cardinal in L violates the Souslin Hypothesis (except for weakly compact cardinal) and almost every cardinal in L satisfies Kurepa Hypothesis.

The next subject treated is Model Theory in L. The main problems considered are two-cardinal problems: Assume that the countable theory T has a model of cardinality κ where the cardinality of some distinguished unary predicate is λ (this model is said to be of type (κ, λ)). Under what conditions can we get a model of T of type (κ', λ')? In the 1-gap case (where k and k' are the successor cardinals of λ and λ' respectively) the problem is almost settled positively if one assumes the generalized continuum hypothesis (Vaught, Kiesler, Chang). The only case missing is when λ' is singular. In Chapter 12 we get rid of this restriction and prove in L the 1-gap theorem for every λ'.

The 2-gap case (where κ and κ' are two cardinals apart from λ and λ' respectively) the situation without $V=L$ is less decided and even GCH and λ' regular can not guarantee the existence of a (κ', λ') model (Silver). But again $V=L$ implies the existence of (κ', λ') model of T using the existence of a combinatorial creature named morass (well deserving its name). That there are morasses, in L is shown in Chapter 13, and in Chapter 14 morasses are used to get the 2-gap result in L. Similar arguments by sinking deeper into the morasses can give an n-gap two cardinal result in L. These results (due to Jensen, of course) set a record in their technical subtlety and it is the first time they are published anywhere. Devlin is doing an important service by publishing them.

The book now explores the implications of the existence of large cardinals on L. The most remarkable fact is that L is inconsistent with very large cardinals (due originally to D. Scott) and if we assume the existence of these cardinals $V=L$ is very badly violated. Again, one of the theorems, of Silver this time, is published here for the first time.

The book concludes by a study of relative constructibility and by showing that the class of sets constructible from a given set is similar in many respects to L and more so if we consider the sets constructible from a normal measure on a measurable cardinal. (In particular, we get a Souslin and Kurepa tree in such a universe.) The Herbacek-Vopenka Theorem, claiming that if there exists a strongly compact cardinal then the universe is not even constructible from a set, is proved.

The aims of the Springer-Verlag Lecture Notes in Mathematics states that "The timeliness of a manuscript is more important than its form, which may be unfinished or tentative." Devlin did not use this option: the standard of exposition in this book is high, and the presentation is very coherent and clear. Though there are places (like the definition of the projection) where more intuitive motivation is highly desirable, the book is an important source for any mathematician seriously interested in the subject.

Reference

1. Gian-Carlo Rota, *Review of mathematical thought from ancient to modern times*, Bull. Amer. Math. Soc. **80** (1974), 805.

MENACHEM MAGIDOR

Analyse convexe et problèmes variationelles, by I. Ekeland and R. Teman, Dunod, Gauthier-Villars, Paris 1974, ix+340 pp.

This book gives a systematic exposition of the modern theories of the calculus of variations and optimal control. Of course the theory of convex sets and functions plays a very important role and the book begins with an elegant exposition of the theory of convex functions. A relatively new notion is that of the *polar* (or *conjugate*) *function* of a given (usually convex) function. The