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ON THE R-FORMS OF CERTAIN ALGEBRAIC VARIETIES

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In this note we determine explicitly the inequivalent models over **R** of certain varieties $U = \Gamma\backslash H^n$ where Γ is a unit group of a totally indefinite quaternion algebra over a totally real number field k, $|k : \mathbf{Q}| = n$, and H = upper half-plane. For each model defined over **R** we give a formula for the number of connected components of the manifold of real points of U.

1. Let A be a totally indefinite division quaternion algebra over a totally real number field k, $\mathfrak{O}$ a maximal order in A, and $\Gamma = \{\gamma \in \mathfrak{O}^\times$ with reduced norm $\nu(\gamma) = 1\}$. We fix an isomorphism λ: $A_{\mathbf{R}} = A \otimes_{\mathbf{Q}} \mathbf{R} \simeq M_2(\mathbf{R})^n$, $n = |k : \mathbf{Q}|$; then $\lambda(\Gamma \otimes 1) \subset \mathrm{SL}_2(\mathbf{R})^n$ and thus $\Gamma/\pm 1$ acts properly discontinuously on H^n = product of n copies of the upper half-plane via fractional linear transformations. Under certain assumptions on A, $\Gamma/\pm 1$ will act without fixed points so that $U = \Gamma\backslash H^n$ will be a compact complex manifold. It is well known that such U are imbeddable as nonsingular complex projective algebraic varieties.

A real model of U is a pair (U', φ) consisting of a nonsingular projective variety $U' \subset \mathbf{P}^N(\mathbf{C})$ defined over **R** and a biholomorphic map φ: $U \longrightarrow U'$. Two real models are equivalent if there exists a biregular isomorphism f: $U_1' \longrightarrow U_2'$ with f defined over **R**. An equivalence class of real models will be called an **R**-form of U. To each real model (U', φ) of U we associate an antiholomorphic involution ρ: $U \longrightarrow U$ by the formula $\rho(x) = \varphi^{-1}(\overline{\varphi(x)})$. We call the points $x \in U$ such that $x = \rho(x)$ the real points of the model (U', φ). The following is well known:

LEMMA 1. *The* **R**-*forms of U are in one-to-one correspondence with the* $\mathrm{Aut}^h(U)$ *conjugacy classes of antiholomorphic involutions on U. Here* $\mathrm{Aut}^h(U)$ = *biholomorphic automorphisms of U.*

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2. Now we have an isomorphism $\mathrm{Isom}(U) \simeq \mathrm{Aut}(A, \Gamma)/\Gamma$ where $\mathrm{Aut}(A, \Gamma)$ = subgroup of $\mathrm{Aut}(A)$ preserving Γ. Notice that if S = the set of primes p of k at which A_p is division and $\mathrm{Aut}(k, S)$ is the subgroup of $\mathrm{Aut}(k)$ which perserves the set S, then

$$1 \longrightarrow A^x/k^x \longrightarrow \mathrm{Aut}(A) \longrightarrow \mathrm{Aut}(k, S) \longrightarrow 1$$

is exact. Certain subsequences of this describe $\mathrm{Isom}(U)$ and $\mathrm{Aut}^h(U)$.

THEOREM I. (1) $\mathrm{Isom}(U) \supset \mathrm{Aut}^h(U)$ *are described by*

$$1 \longrightarrow I(k)_2 \times Z_2^m \times U_k/U_k^2 \longrightarrow \mathrm{Isom}(U) \longrightarrow \mathrm{Aut}(k, S, \Gamma) \longrightarrow 1,$$

$$1 \longrightarrow I(k)_2' \times Z_2^{m'} \times U_k^+/U_k^2 \longrightarrow \mathrm{Aut}^h(U) \longrightarrow \mathrm{Aut}^h(k, S, \Gamma) \longrightarrow 1$$

where the subgroups are isomorphic to $N_A*(\Gamma)/k^\times\Gamma$ *and* $N_{A_+^\times}(\Gamma)/k^\times\Gamma$ *respectively.* $I(k)_2$ = *the two torsion in the ideal class group* $I(k)$ *of* k, $I(k)_2'$ = *image in* $I(k)$ *of the two torsion of the narrow ideal class group* $I_1(k)$, U_k *(resp.* U_k^+*) = units (resp. totally positive units) of* k, *and the elements of* Z_2^m *(resp.* $Z_2^{m'}$*) correspond to tuples* $(e_1, \cdots, e_{|S|})$, $e_i = 0, 1$, *such that* $[\Pi_{i=1}^{|S|} p_i^{e_i}] \in I(k)^2$ *(resp.* $[\Pi_{i=1}^{|S|} p_i^{e_i}] \in I_1(k)^2$*),* $p_i \in S$. *Finally, there exists a certain* 1*-cocycle* $\tau \longrightarrow t_{\tau,\mathfrak{G}}$ *in* $Z^1(\mathrm{Aut}(k, s), I_1(k)/I_1(k)^2 I_1^S(k))$ *where* $I_1^S(k)$ *is the subgroup of* $I_1(k)$ *generated by the primes of* S *such that* $\mathrm{Aut}^h(k, S, \Gamma) = \{\tau \in \mathrm{Aut}(k, S)$ *such that* $t_{\tau,\mathfrak{G}} = 1\}$. *A similar* 1*-cocycle with values in* $I(k)/I(k)^2 I^S(k)$ *describes* $\mathrm{Aut}(k, S, \Gamma)$.

(2) *If* $\beta \in N_{A^\times}(\Gamma)$ *is any element of totally negative reduced norm–such elements exist by strong approximation–then the coset* $\beta N_{A_+^\times}(\Gamma)/k^\times\Gamma$ *in* $N_{A^\times}(\Gamma)/k^\times\Gamma$ *consists of all elements of this group which correspond to antiholomorphic involutions in* $\mathrm{Isom}(U)$. *Finally, since* $N_{A^\times}(\Gamma)/k^\times\Gamma$ *is abelian, any two elements* x *and* x' *in* $\beta N_{A_+^\times}(\Gamma)/k^\times\Gamma$ *correspond to* $\mathrm{Aut}^h(U)$ *conjugate involutions* $\iff \exists\, \tau \in \mathrm{Aut}^h(k, S, \Gamma)$ *such that* $x' = x^\tau$.

COROLLARY. *If we let* $[\beta]$ *denote the element of* $I(k)_2 \times Z_2^m \times U_k/U_k^2 \simeq N_{A^\times}(\Gamma)/k^\times\Gamma$ *corresponding to* β, *then the set*

$$[\beta]\, I(k)_2' \times Z_2^{m'} \times U_k^+/U_k^2/\mathrm{Aut}^h(k, S, \Gamma)$$

is in one-to-one correspondence with a subset of the **R***-forms of* U.

REMARK. The theorem classifies only those **R**-forms coming from involutions which do not permute the factors of H^n; of course if $\mathrm{Aut}(k, S)$ is trivial, these are all.

3. We now restrict ourselves to those antiholomorphic involutions and

R-forms arising from the elements of $N_{A^\times}(\Gamma)/k^\times\Gamma$. We have the following criterion:

LEMMA 2. *An antiholomorphic involution ρ of U has a fixed point $\Longleftrightarrow$ $\exists\, \beta \in N_{A^\times}(\Gamma)$ representing it such that $\beta^2 \in k^\times$. More conveniently, suppose $\alpha \in N_{A^\times}(\Gamma)$ is any element representing ρ; then ρ has a fixed point on $U \Longleftrightarrow$ the quadratic extension $k(\sqrt{-\nu(\alpha)}) = K$ is imbeddable in A.*

As for the number of connected components of the real points in the case where K is imbeddable, an easy covering space argument together with a computation analogous to that used by Eichler [1] and Shimizu [3] to compute the trace of Hecke operators yields an explicit formula.

THEOREM II. *The number of connected components of the fixed point set of $\rho = 2^{s-t-1}h(k)^{-1}\Sigma_{\mathfrak{D}_i} h(\mathfrak{D}_i)|U_k: N_{K/k}(\mathfrak{D}_i^\times)|$ where $s = |S|$, t = number of primes of S which are ramified in K/k, $h(k)$ = class number of k, $\mathfrak{D}_i$ runs over all orders of K which occur as $\mathfrak{D} = \psi^{-1}(\psi(K) \cap \mathfrak{O})$ for ψ an imbedding of K into A which satisfies $\psi(\sqrt{-\nu(\alpha)}) \in N_{A^\times}(\Gamma)$. $h(\mathfrak{D}_i)$ = class number of the order $\mathfrak{D}_i$, $\mathfrak{D}_i^\times$ = units of $\mathfrak{D}_i$, and $N_{K/k}$ is the usual norm from K to k.*

Notice that the sum is finite because each order of the above sort must contain the order $\mathfrak{O}'_K = \mathfrak{O}_k + \mathfrak{O}_K \cap (k^\times\sqrt{-\nu(\alpha)})$, $\mathfrak{O}_K$ = *maximal order of* K.

I have recently discovered that Professor Shimura is working on similar questions.

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