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NECESSARY AND SUFFICIENT CONDITIONS FOR DETERMINING A HILL'S EQUATION FROM ITS SPECTRUM[1]

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A Hill's equation is an equation of the form

$$y'' + [\lambda - q(z)]y = 0. \tag{1}$$

We assume $q(z + \pi) = q(z)$, where $q(z)$ is integrable over $[0, \pi]$. Without loss of generality, it is customary to assume that $\int_0^\pi q(z)\,dz = 0$. The discriminant of (1) is defined by $\Delta(\lambda) = y_1(\pi) + y_2'(\pi)$ where y_1 and y_2 are solutions of (1) satisfying $y_1(0) = y_2'(0) = 1$ and $y_1'(0) = y_2(0) = 0$.

The set of values of λ for which $|\Delta| > 2$ consists of a finite or an infinite number of finite disjoint intervals and one infinite interval. These intervals are called instability intervals, since (1) has no solution which is bounded for all real z in these intervals. When $|\Delta| < 2$, all solutions of (1) are bounded for all real z and the corresponding intervals are called stability intervals. Pertinent information about stability and instability intervals of (1) can be found in Magnus and Winkler [1].

In [2] it was proved that if $q(z)$ is real and integrable, and if precisely n finite instability intervals fail to vanish, then $q(z)$ must satisfy a differential equation of the form

$$q^{(2n)} + H(q, q', \cdots, q^{(2n-2)}) = 0 \tag{2}$$

where H is a polynomial of maximal degree $n + 2$. Explicit expressions of this equation are displayed in [2] and [3] for the cases $n = 0, 1$ and 2.

For an infinite class of Korteweg-deVries equations of the form

$$q_t = K_n(q, q_z, \cdots, \partial^{2n+1}q/\partial z^{2n+1}) \quad (n = 0, 1, 2, \cdots),$$

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Lax [4] has found that a periodic function q satisfying

$$K_n(q, \cdots, \partial^{2n+1}q/\partial z^{2n+1}) = 0 \tag{3}$$

requires (1) to have no more than n finite instability intervals. Explicit computations in [2] have shown (2) and (3) to be equivalent for the cases $n = 0, 1$ and 2. These results have now been extended to show that (2) and (3) are equivalent for all values of n. Hence we have necessary and sufficient conditions which the periodic potential function $q(z)$ must satisfy when n finite instability intervals of (1) fail to vanish.

The proof of this result is accomplished by comparing an asymptotic expression of the solution of the related problem

$$\begin{cases} u'' + [\lambda - q(z + t)]\ u = 0; \ t \text{ real, arbitrary} \\ u(0) = 0, u'(0) = 1 \end{cases}$$

at $z = \pi$ with $y_2(\pi)$ and by assuming Hochstadt's result [5] that $q(z)$ is infinitely differentiable when n finite instability intervals fail to vanish. The details will appear in a later publication.

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