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BOOK REVIEW

Primideale in Einhüllenden auflösbarer Lie-algebren, by Walter Borho, Peter Gabriel and Rudolph Rentschler, Springer-Verlag, Berlin, 182 pp., \$7.00

For a proper appreciation of this work, it should be kept in mind that Lie algebras have a significance reaching beyond the domain of algebra, because they play such an important role in the theory of Lie groups. Thus, classical Lie algebra theory is strongly dominated by the fact that the finite-dimensional analytic representations of a simply connected analytic group are identifiable with the finite-dimensional representations of its Lie algebra. In the theory of infinite-dimensional representations, the connection with Lie algebra representations is somewhat tenuous, but it is nevertheless at the core of the major advances made in that theory during the last 30 years.

Although the theory presented here is purely algebraic, it bears the stamp of Lie group theory. Its origin lies in Kirillov's classification of the irreducible unitary representations of simply connected nilpotent real analytic groups (1961). Kirillov's methods and results were extended toward the solvable case through a variety of intermediate stages. The most important references for these developments are: L. Auslander and B. Kostant, *Polarization and unitary representations of solvable Lie groups*, Invent. Math. **14** (1971), 255–354, and the monograph (Société Mathématique de France) *Représentations des groupes de Lie résolubles* by P. Bernat, N. Conze, M. Duflo, M. Lévy-Nahas, M. Rais, P. Renouard, and M. Vergne (Dunod, Paris, 1972). The main result is that the irreducible unitary representations of the solvable groups of type I can all be obtained as induced representations from 1-dimensional representations of suitable subgroups (this generalization of Kirillov's result is due to L. Auslander and B. Kostant).

In a series of papers, beginning in 1963, J. Dixmier developed the corresponding technique for the study of the universal enveloping algebra of a solvable (complex) Lie algebra. Dixmier's results were completed only quite recently, notably by K. Duflo (1970) and R. Rentschler (1973), and the present book is the first self-contained (and improved) exposition of the full theory, for the solvable case. A natural companion work, containing also the very different theory for the semisimple case, is

J. Dixmier's *Algèbres enveloppantes* (Gauthier-Villars, Paris, 1974), which has just appeared.

It seems that the variety of all simple modules for the universal enveloping algebra of a solvable Lie algebra is so vast as to escape classification. However, it turns out that their annihilators in the universal enveloping algebra, i.e., the primitive ideals, *can* be classified. The principal result accomplishing this is that the primitive ideals are in bijective correspondence with the orbits in the linear dual of the Lie algebra with respect to the action of the algebraic hull of the adjoint group. As in the case of groups, this comes about through a construction of induced modules, starting from a linear functional on the Lie algebra. The proof of the bijectiveness of the attendant map of the orbit space into the space of the primitive ideals is difficult, and it is completed only near the end of the book.

Quite apart from its connection with group representations, the study of the universal enveloping algebra presented here is of great ring-theoretical interest. Indeed, it is an impressive model of a most successful application of noncommutative localization and prime ideal theory, whose general features are developed with admirable elegance and efficiency in the first chapter. The remaining three chapters deal specifically with the enveloping algebra. In addition to the main result cited above, there is provided a deep insight into the prime ideal structure and the corresponding residue class algebras. A detailed description of the results would involve too many technicalities to be feasible here. A sample illustrating the nature of these structural results is a proof of the Gelfand-Kirillov conjecture for algebraic Lie algebras in the solvable case (A. Joseph, J. McConnel, W. Borho, 1973), according to which the total quotient division algebra is isomorphic with that of a Weyl algebra over a finitely generated commutative algebra.

Finally, it is worthy of note that this book has been written with very much more care than is suggested by the form of publication (photographed typescript). It is a highly polished, tightly organized work, written with the full masterly control of important contributors to this area of research. The content is of strong current interest to those concerned primarily with Lie groups, as well as to ring theorists.

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