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CHARACTERIZATIONS OF KNOTS AND LINKS

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Although there are inequivalent classical knots with isomorphic groups, J. Simon recently characterized each knot type by a group: the free product of two, suitably chosen, cable-knot groups [4]. In this paper, we announce other characterizations, both algebraic and geometric, that are more direct, cover links as well as knots, and yield characterizations of amphicheiral knots.

In Section 1, we give preliminaries and state two lemmas. In Section 2, we outline the proof of the new characterizations, the combined results of the papers [7] and [8], which contain detailed proofs.

1. **Preliminaries.** Throughout this work, the three-sphere S^3 has a fixed orientation; all maps are piecewise linear; all submanifolds, subpolyhedra; and all regular neighborhoods, at least second regular. If L is a link in S^3, then $\{L\}$ denotes the (ambient) isotopy type of L; the symbol L^*, the mirror image of L.

Let $L\ (= K_1 \cup \cdots \cup K_\mu)$ be a link in S^3. For each of $i = 1, \cdots, \mu$, let V_i be a closed regular neighborhood of K_i and let $\mathcal{K}_i$ be a knot in $\operatorname{Int} V_i$. We assume that $V_i \cap V_j = \varnothing$ when $i \neq j$. We also assume that V_i has order greater than zero with respect to $\mathcal{K}_i$ $(i = 1, \cdots, \mu)$. We set $R(L) = \mathcal{K}_1 \cup \cdots \cup \mathcal{K}_\mu$, and we call $R(L)$ a revision of L.

Let (ρ, η) be a pair of integers; ρ, arbitrary; $\eta = \pm 2$. For each of $i = 1, \cdots, \mu$, let Y_i denote a singular disk that has exactly one clasping singularity, that belongs to $\operatorname{Int} V_i$, and that has ρ as its twisting number, η as its intersection number with its boundary, and K_i as its diagonal; see [3, Section 20, p. 232]. If $\mathcal{K}_i$ is the boundary of Y_i, we shall denote $R(L)$ by $D(L; \rho, \eta)$ and call it the (ρ, η)-double of L. Note that $D(K_i; \rho, \eta)\ (= \mathcal{K}_i)$ is the (ρ, η)-double of K_i.

LEMMA 1.1. *Let $L (= K_1 \cup \cdots \cup K_\mu)$ be a link in S^3, and let $R(L)$ be any revision of L. Then L is splittable if and only if $R(L)$ is splittable.*

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LEMMA 1.2. *Let L and L' be links in S^3, and let (ρ, η) and (ρ', η') be pairs of integers; ρ and ρ', arbitrary; η and η', in $\{2, -2\}$. If $\{L\} = \{L'\}$, if $\rho = \rho'$, and if $\eta = \eta'$, then $\{D(L; \rho, \eta)\} = \{D(L'; \rho', \eta')\}$. Conversely, if $\{D(L; \rho, \eta)\} = \{D(L'; \rho', \eta')\}$, then $\{L\} = \{L'\}$; furthermore, $\rho = \rho'$ and $\eta = \eta'$, unless*

(1) *some component K_i of $D(L; \rho, \eta)$ is a maximal unsplittable sublink of $D(L; \rho, \eta)$ and*

(2) *K_i is either the trivial or the figure-eight knot.*

In particular, $\{D(L; \rho, \eta)\} = \{D(L'; \rho', \eta')\}$ if and only if $(\{L\}, \rho, \eta) = (\{L'\}, \rho', \eta')$, provided that the number of components of L is $\geqslant 2$ and that $D(L; \rho, \eta)$ is unsplittable.

Proofs of Lemmas 1.1 and 1.2 are nontrivial and interesting, but even summaries of these proofs are too long for this announcement.

2. **The characterizations.** Let $D(L; \rho, \eta)$ be the (ρ, η)-double of a link $L(= K_1 \cup \cdots \cup K_\mu)$, and for each of $i = 1, \cdots, \mu$, let W_i be a closed regular neighborhood of K_i. We assume that $W_i \subset \operatorname{Int} V_i$ $(i = 1, \cdots, \mu)$ and set $C^3(L; \rho, \eta) = S^3 - \operatorname{Int}(W_1 \cup \cdots \cup W_\mu)$.

THEOREM 2.1. *Let L and L' be links in S^3 and let (ρ, η) be a pair of fixed integers; ρ, arbitrary; $\eta = \pm 2$. Then L and L' belong to the same ambient isotopy type if and only if $\pi_1(C^3(L; \rho, \eta)) \approx \pi_1(C^3(L'; \rho, \eta))$.*

COROLLARY 2.2. *The links L and L' belong to the same ambient isotopy type if and only if $C^3(L; \rho, \eta) \cong C^3(L'; \rho, \eta)$.*

PROOF. The necessity follows from Lemma 1.2; the sufficiency, from Theorem 2.1.

COROLLARY 2.3. *A knot K is amphicheiral if and only if $\pi_1(C^3(K; \rho, \eta)) \approx \pi_1(C^3(K^*; \rho, \eta))$. Furthermore, K is amphicheiral if and only if $C^3(K; \rho, \eta) \cong C^3(K^*; \rho, \eta)$.*

OUTLINE OF THEOREM 2.1'S PROOF. Lemma 1.2 immediately establishes the necessity; to prove the sufficiency, we assume, henceforth, that $\pi_1(C^3(L; \rho, \eta)) \approx \pi_1(C^3(L'; \rho, \eta))$. This hypothesis, Theorem (27.1) of [2], and the uniqueness of a group's decomposition as a free product combine to establish a bijective correspondence between the maximal unsplittable sublinks $L_1, \cdots, L_m$ of L and the maximal unsplittable sublinks $L'_1, \cdots, L'_{m'}$ of

L' such that $m = m'$ and $\pi_1(C^3(L'_{k_j}; \rho, \eta)) \approx \pi_1(C^3(L_j; \rho, \eta))$ $(j = 1, \cdots, m)$. Because the collection $\{\{L_1\}, \cdots, \{L_m\}\}$ determines $\{L\}$, we shall assume not only that $\pi_1(C^3(L; \rho, \eta)) \approx \pi_1(C^3(L'; \rho, \eta))$, but also that each of L and L' and, hence, each of $D(L; \rho, \eta)$ and $D(L'; \rho, \eta)$ is unsplittable. Note that the number of components in each of L, L', $D(L; \rho, \eta)$, and $D(L'; \rho, \eta)$ is μ. The detailed proofs given in [7] and [8] cover the cases $\mu = 1$ and $\mu > 1$ separately; because of spatial limitations here, however, we shall assume, from now on, that $\mu > 1$.

Set $T_i = \partial V_i$, set $C' = C^3(L'; \rho, \eta)$, and set $C = C^3(L; \rho, \eta)$. Also, set $M = S^3 - \mathrm{Int}(V_1 \cup \cdots \cup V_\mu)$, set $M_i = C - \mathrm{Int}\ V_i$, and set $\Lambda_i = V_i - \mathrm{Int}\ W_i (i = 1, \cdots, \mu)$. Lemma 1.1 implies that each M_i is boundary irreducible; each Λ_i is boundary irreducible because, if μ_i is a meridian of V_i, then $\mu_i \cup K_i$ is unsplittable; therefore, $\pi_1(C) \approx \pi_1(M_i) *_{\pi_1(T_i)} \pi_1(\Lambda_i)$ $(i = 1, \cdots, \mu)$. Because there is only one link whose group is $|x, y : xyx^{-1}y^{-1}|$, the group $\pi(M_i) *_{\pi_1(T_i)} \pi_1(\Lambda_i)$ is nontrivial as a free product with amalgamation.

Because $\pi_1(C') \approx \pi_1(C)$ and because each of C' and C is aspherical, there exists a homotopy equivalence $f: C' \longrightarrow C$. Hence, by [5, Lemma 1.1, p. 506], there exists a mapping $g: C' \longrightarrow C$ with the following properties:

(1) $g \cong f$; (2) g *is transverse with respect to* $T (= T_1 \cup \cdots \cup T_\mu)$; (3) $g^{-1}(T)$ *is a compact orientable surface properly imbedded in* C'; (4) *if* F *is any component of* $g^{-1}(T)$, *then* $\ker(\pi_j(F) \longrightarrow \pi_j(C')) = 1$ $(j = 1, 2)$.

We divide the remainder of this outline into seven parts.

1. *For each of* $i = 1, \cdots, \mu$, *the space* $g^{-1}(T_i)$ *is not empty.*

If $g^{-1}(T_i) = \emptyset$, either $g_*(\pi_1(C')) \subseteq \pi_1(M_i)$ or $g_*(\pi_1(C')) \subseteq \pi_1(\Lambda_i)$. Thus, because $\pi_1(C)$ is a nontrivial free product with amalgamation, $g_*(\pi_1(C'))$ is a *proper* subgroup of $\pi_1(C_i)$. But g_* is an isomorphism; therefore, $g^{-1}(T_i) \neq \emptyset$.

LEMMA 2.4. *Any properly imbedded, incompressible annulus* A *in* C *is boundary parallel.*

OUTLINE OF PROOF. One must establish four points: (a) $\partial A \subset \partial W_j$ for some j; (b) there is an isotopy of C moving A into $\mathrm{Int}\ V_j$; (c) there is an annulus A_1 on ∂W_j and there is a solid torus $X \subset \Lambda_j$ such that $\partial X = A \cup A_1$; (d) A is parallel to A_1 in X.

2. *We can assume that each component* F *of* $g^{-1}(T_i)$ *and, hence, each component of* $g^{-1}(T)$ *is a torus that is not boundary parallel.*

The group $\pi_1(F)$ must be isomorphic to a subgroup of $\pi_1(T_i)$; thus, F is either a 2-sphere, a disk, an annulus, or a torus. Property (4) implies that $\pi_2(F) = 0$; hence, F is not a 2-sphere. If F is a boundary-parallel annulus or torus, or if F is a disk, then we can easily replace g by a map g': $C' \longrightarrow C$ satisfying the properties (1) through (4) and the property that $g'^{-1}(T_i)$ has fewer components than $g^{-1}(T_i)$. Lemma 2.4 applies to C' and implies that F is not an annulus.

3. *For each of* $i = 1, \cdots, \mu$ *and for each component* F *of* $g^{-1}(T_i)$, *we can assume that* $g|F$ *is a homeomorphism.*

Because $g|F$ is homotopic to a covering map $k\colon F \longrightarrow T_i$ [6, Lemma 1.4.3, p. 61] and because g is transverse with respect to T_i, there is a homotopy $\{h_t\}$ $(0 \leqslant t \leqslant 1)$ of g such that $k = h_1|F$. Now h_{1*} is an isomorphism and $h_{1*}(\pi_1(F)) = \pi_1(T_i)$ [1, Theorem 1, p. 575]. Therefore, k is a homeomorphism.

4. *For each of* $i = 1, \cdots, \mu$, *we can assume that* $g^{-1}(T_i)$ *is connected, and hence, that* $g^{-1}(T)$ *has exactly* μ *components,* $T'_1, \cdots, T'_\mu$, *with* $g^{-1}(T_i) = T'_i$.

A variation of J. R. Stallings' "binding tie" argument couched in an inductive proof establishes 4.

5. (a) *There are mutually disjoint, solid tori,* $V'_1, \cdots, V'_\mu$, *such that* $\partial V'_i = T'_i$ *and such that* $W'_i \subset \operatorname{Int} V'_i$ $(i = 1, \cdots, \mu)$ *for a suitable change in the subscripts of* $W'_1, \cdots, W'_\mu$.

(b) *Setting* $M' = S^3 - \operatorname{Int}(V'_1 \cup \cdots \cup V'_\mu)$, *we have* $M' \cong M$, *and we can assume that* $g|M'$ *is a homeomorphism.*

There are manifolds M'_i and Λ'_i such that $C' = M'_i \cup_{T'_i} \Lambda'_i$, such that $g_*(\pi_1(M'_i)) = \pi_1(M_i)$, and such that $g_*(\pi_1(\Lambda'_i)) = \pi_1(\Lambda_i)$. Consequently, $\partial\Lambda'_i = T'_i \cup \partial W'_i$ for some W'_j with i in place of j. Set $V'_i = \Lambda'_i \cup W'_i$.

Let D'_i be a singular disk that spans K'_i, that has exactly one clasping singularity, and that misses K'_j when $j \neq i$. We move D'_i into $\operatorname{Int} V'_i$ by an ambient isotopy leaving $D(L'; \rho, \eta)$ fixed. Then $D'_i \cup W'_i$ has a closed regular neighborhood N'_i that is a solid torus, that belongs to $\operatorname{Int} V'_i$, and that has the diagonal of D'_i as a core. Because $\partial N'_i$ is not parallel to $\partial W'_i$ and because–and the proof is tortuous–every properly imbedded, incompressible torus in Λ'_i is boundary parallel, $\partial N'_i$ is parallel to T'_i. Therefore, T'_i is compressible in V'_i, and so V'_i is a solid torus. Furthermore, $\Lambda'_i \cong N'_i - \operatorname{Int} W'_i \cong \Lambda_i$, and there are faithful homeomorphisms $(V'_i, K'_i) \longrightarrow (N'_i, K'_i) \longrightarrow (V_i, K_i)$.

Now $g(M'_1) \subseteq M_1$, the homomorphism $(g|M'_1)_*$ is an isomorphism, $M'_1 = (M'_1 - \text{Int } V'_2) \cup_{T'_2} \Lambda'_2$, and $M_1 = (M_1 - \text{Int } V_2) \cup_{T_2} \Lambda_2$; therefore, we have

$$\pi_1(M'_1 - \text{Int } V'_2) \approx \pi_1(M_1 - \text{Int } V_2).$$

Arguing inductively, we obtain $\pi_1(M') \approx \pi_1(M)$. Each $g|T'_i$ is a homeomorphism. Therefore, there exists a homotopy from g to a map $g'\colon C' \longrightarrow C$ such that $g'|M'$ is a homeomorphism and such that the homotopy is constant on $C' - \text{Int } M'$ [6, Theorem 6.1, p. 77].

6. (a) *For each of* $i = 1, \cdots, \mu$, *we have* $\Lambda'_i \cong \Lambda_i$, *and there is a faithful homeomorphism* $(V'_i, K'_i) \longrightarrow (V_i, K_i)$.

(b) *If* k_i *is a core of* V'_i, *then* $\{k_1 \cup \cdots \cup k_\mu\} = \{L'\}$.

We proved **6**(a) in the proof of **5**(a); Lemma 1.2 implies **6**(b).

7. $\{L'\} = \{L\}$.

Set $G_1 = \pi_1(\Lambda'_i)$ and $G_2 = \pi_1(\Lambda_i)$. We have

$$(*) \qquad \begin{aligned} G_j = |u_j, z_j, x_j \colon & z_j u_j z_j^{-1} u_j^{-1}, \\ & u_j = x_j u_j^{\rho} z_j^{-1} x_j^{-1} z_j u_j^{-\rho} x_j^{-1} u_j^{\rho} z_j^{-1} x_j z_j u_j^{-\rho}|. \end{aligned}$$

The pair (u_1, z_1) is a meridian-longitude pair for V'_i; the pair (u_2, z_2), a meridian-longitude pair for V_i. Because $g|M'$ is a homeomorphism, we have $g_*(z_1) = u_2^r z_2^v$ and $g_*(u_1) = u_2^{\pm 1} z_2^q$. If $\mathfrak{a} : G_2 \longrightarrow G_2/G'_2$, then application of $\mathfrak{a} g_*$ to the second relation of G_1 in $(*)$ shows that $q = 0$; hence, $k_1 \cup \cdots \cup k_\mu$ and L are equivalent. If, however, $\{k_1 \cup \cdots \cup k_\mu\} \neq \{L\}$, one can construct certain factor groups of G_1 and G_2 that must be both isomorphic and nonisomorphic. This concludes the outline of the Theorem's proof.

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