BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 80, Number 6, NOVEMBER 1974

SPECTRAL THEORY FOR BOUNDARY VALUE PROBLEMS FOR ELLIPTIC SYSTEMS OF MIXED ORDER

BY GIUSEPPE GEYMONAT AND GERD GRUBB

Communicated by Robert Seeley, May 28, 1974

Introduction. For a closed, densely defined linear operator T in a Hilbert space H, we define the essential spectrum ess sp T as the complement in $\mathbf{C}$ of the set of λ for which $T - \lambda$ is a Fredholm operator (with possibly nonzero index). Recall (cf. Wolf [7]) that $\lambda \in$ ess sp T if and only if either $T - \lambda$ or $T^* - \overline{\lambda}$ has a singular sequence, i.e. a sequence $u_k \in H$ with $\|u_k\| = 1$ for all k, $(T - \lambda)u_k \longrightarrow 0$ (or $(T^* - \overline{\lambda})u_k \longrightarrow 0$) in H, but u_k having no convergent subsequence in H. ess sp T is closed and invariant under compact perturbations of T, and contains the accumulation points of the eigenvalue spectrum.

Let $\overline{\Omega}$ be an n-dimensional compact C^∞ manifold with boundary Γ and interior $\Omega = \overline{\Omega}\backslash\Gamma$. It is well known that when A is a properly elliptic operator on $\overline{\Omega}$ of order $r > 0$, the L^2-realization $A_B : u \longmapsto Au$ with domain $D(A_B) = \{u \in L^2(\Omega) | Au \in L^2(\Omega), Bu|_\Gamma = 0\}$, defined by a boundary operator B that covers A (i.e. $\{A, B\}$ defines an elliptic boundary value problem), has ess sp $A_B = \emptyset$.

However, when A is a *system* of *mixed order*, elliptic in the sense of Douglis and Nirenberg (cf. [1]), ess sp A_B can be nonempty even when $\{A, B\}$ is elliptic with smooth coefficients and $\overline{\Omega}$ is compact. We study this phenomenon for a class of Douglis-Nirenberg systems of nonnegative order, determine the essential spectrum, and find the asymptotic behavior of the discrete spectrum at $+\infty$ for the selfadjoint lower bounded realizations.

Examples of the systems we consider are: The linearized Navier-Stokes operator and certain systems stemming from nuclear reactor analysis. A preliminary, less advanced account of the theory was given in [5].

1. Preliminaries.

1.1. For q integer > 1 there is given a set of integers $m_1 \geqslant m_2 \geqslant$

AMS (MOS) subject classifications (1970). Primary 35J45, 47B30, 35P20.

$\cdots \geqslant m_{q'} > m_{q'+1} = \cdots = m_q = 0$; we assume $1 \leqslant q' < q$ and denote $\max m_s = m$. Let $\mathsf{A} = (\mathsf{A}_{st})_{s,t=1,\cdots,q}$ be a $q \times q$-matrix of differential operators of orders $m_s + m_t$ on $\overline{\Omega}$. A is assumed *elliptic* (in the sense of [1]), i.e. the principal symbol $\sigma^0(\mathsf{A}) = (\sigma_{m_s+m_t}(\mathsf{A}_{st}))$ has nonzero determinants on $T^*(\overline{\Omega})\backslash 0$. It is useful to single out the zero order part of A by splitting the rows and columns into the first q' and the last $q - q'$ entries:

$$\mathsf{A} = \begin{pmatrix} P & Q \\ R & M \end{pmatrix}; \tag{1.1}$$

here P, Q and R are of positive order, and M is a *multiplication* operator.

1.2. The notation for boundary conditions follows Grubb [4, Chapter 3]: For $u = \{u_1, \cdots, u_q\}$, $\beta^0 u$ denotes the Dirichlet data (the collection of the normal derivatives of each u_t up to order $m_t - 1$), and $\beta^1 u$ denotes the remaining normal derivatives up to orders $m_t + m - 1$, arranged as in [4]; $\beta u = \{\beta^0 u, \beta^1 u\}$ constitute the "Cauchy data". We consider boundary conditions of the form

$$B^{00}\beta^0 u = 0, \qquad B^{10}\beta^0 u + B^{11}\widetilde{\mathfrak{A}}^{01}\beta^1 u = 0, \tag{1.2}$$

where $\widetilde{\mathfrak{A}}^{01}$ is a certain fixed surjective differential operator entering in Green's formula, and the $B^{\cdot\cdot}$ are systems of differential operators of suitable orders (cf. [4], selfadjoint or semibounded realizations necessarily stem from boundary conditions of this form, and it can be shown that boundary problems in general reduce to at least *inhomogeneous* conditions on $\beta^0 u$ and $\widetilde{\mathfrak{A}}^{01}\beta^1 u$, the "reduced Cauchy data"). We assume thoughout that (1.2) *covers* A (satisfies the conditions in [1]).

2. **The case of a manifold without boundary.**

2.1. Assume first that $\overline{\Omega}$ is compact with $\Gamma = \varnothing$, i.e. $\overline{\Omega} = \Omega$. Then A has a parametrix $\widetilde{\mathsf{A}}$, which we split in the same blocks as (1.1):

$$\widetilde{\mathsf{A}} = \begin{pmatrix} \widetilde{P} & \widetilde{Q} \\ \widetilde{R} & \widetilde{S} \end{pmatrix}; \tag{2.1}$$

here $\widetilde{P}, \widetilde{Q}$ and $\widetilde{R}$ are pseudodifferential operators of negative (mixed) order, and $\widetilde{S}$ is a ps.d.o. of order zero. A has only one L^2-realization, which we call A,

$$D(A) = \{u \in L^2(\Omega)^q \mid \mathrm{A}u \in L^2(\Omega)^q\}. \tag{2.2}$$

One finds by use of $\widetilde{\mathrm{A}}$ that $D(A) \subset \Pi_{t=1}^q H^{m_t}(\Omega)$.

THEOREM 2.1. *$\operatorname{ess\,sp} A = \{\lambda \neq 0 \mid \lambda^{-1} \in \operatorname{ess\,sp} \widetilde{S}\} = \{\lambda \neq 0 \mid \lambda^{-1}$ is an eigenvalue for $\sigma^0(\widetilde{S})(x, \xi)$ for some $(x, \xi) \in T^*(\overline{\Omega})\backslash 0\} = \{\lambda \mid \mathrm{A} - \lambda$ is not elliptic$\}$.*

SKETCH OF PROOF. The first identity follows from the fact that

$$\widetilde{\mathrm{A}} = \begin{pmatrix} 0 & 0 \\ 0 & \widetilde{S} \end{pmatrix} + \text{compact operator in } L^2(\Omega)^q. \tag{2.3}$$

In the second identity, the inclusion $\subset$ is immediate; on the other hand, when μ is an eigenvalue for $\sigma^0(\widetilde{S})(x_0, \xi_0)$ with eigenvector $\theta, \widetilde{S} - \mu$ has the singular sequence (in a local coordinate system where $x_0 = 0$)

$$w_k(x) = k^{n/2} v(kx) \exp(i\langle x, k^2 \xi_0\rangle)\theta, \quad k \longrightarrow \infty, \tag{2.4}$$

where $v \in C_0^\infty(\mathbf{R}^n)$ with $v(0) = 1$ and $\|v\|_0 = 1$. Finally, the last identity follows from the equation, valid for $\lambda \in \mathbf{C}$,

$$\det \sigma^0(\mathrm{A} - \lambda) = \det \sigma^0(I - \lambda\widetilde{S}) \det \sigma^0(\mathrm{A}). \tag{2.5}$$

REMARK. $\operatorname{ess\,sp} A$ is *bounded* if and only if P (cf. (1.1)) is elliptic.

2.2. Furthermore, assume now that A is strongly elliptic (i.e., $\sigma^0(\mathrm{A}) + \sigma^0(\mathrm{A})^*$ is positive definite on $T^*(\overline{\Omega})\backslash 0$) and formally selfadjoint. Then, in particular, P is strongly elliptic, so $\operatorname{ess\,sp} A$ is bounded. Since A is unbounded (and selfadjoint, lower bounded), it has a sequence of eigenvalues $\lambda_j^+(A)$ converging to $+\infty$ for $j \longrightarrow \infty$. For large λ, the eigenvalue problem

$$(P - \lambda)v + Qw = 0, \qquad Q^*v + (M - \lambda)w = 0$$

is equivalent with the *nonlinear* problem

$$(P - Q(M - \lambda)^{-1}Q^*)v - \lambda v = 0. \tag{2.6}$$

Here $P - Q(M - \lambda)^{-1}Q^*$ approaches P as λ becomes large, so (2.6) approaches an eigenvalue problem for P in some sense. Indeed, we can show (see [5]):

THEOREM 2.2. *Let* A *be strongly elliptic and formally selfadjoint. The spectrum of* A *on* $]\,\|M\|, +\infty[$ *is a sequence of eigenvalues* $\lambda_1^+(A) \leqslant \lambda_2^+(A) \leqslant \cdots$ *(repeated according to multiplicities) converging to* $+\infty$ *as follows*: $\lambda_j^+(A) \sim \lambda_j(P) \sim cj^{2mq'/n}$ *for* $j \longrightarrow \infty$, *where* c *is a constant determined from* $\sigma^0(P)$, *and* $a_j \sim b_j$ *for* $j \longrightarrow \infty$ *means* $a_j/b_j \longrightarrow 1$ *for* $j \longrightarrow \infty$.

3. The case of a manifold with boundary.

3.1. Consider now the case where $\Gamma \neq \emptyset$. We may assume that $\bar{\Omega}$ is smoothly imbedded in an n-dimensional C^∞ manifold Σ without boundary, A extended to an elliptic operator, also called A, on Σ. Let $\widetilde{A}$ be a parametrix of A on Σ (a properly supported $q \times q$-matrix pseudodifferential operator satisfying $\sigma(A\widetilde{A}) = \sigma(\widetilde{A}\,A) = I$).

Let A_B be the L^2-realization of A in Ω determined by (1.2):

$$D(A_B) = \{u \in L^2(\Omega)^q \mid Au \in L^2(\Omega)^q, \text{(1.2) holds}\}. \tag{3.1}$$

It is well known that $A_B - \lambda$ is Fredholm *as an operator from* $\Pi_{t=1}^q H^{m+m_t}(\Omega)$ *to* $\Pi_{t=1}^q H^{m-m_t}(\Omega)$ (bounded operator!) if and only if (1.2) covers $A - \lambda$. For *unbounded* realizations, one may have the Fredholm property without covering (we have examples where the unbounded realization *in* $\Pi_{t=1}^q H^{m-m_t}(\Omega)$ falls into the uniformly nonelliptic class of Vaĭnberg and Grušin [6]). However, we have found for the *L^2-realization*:

THEOREM 3.1. *Let* $\omega = \{\lambda \mid A - \lambda$ *is not elliptic on* $\bar{\Omega}\}$, $\omega_B = \{\lambda \mid \lambda \notin \omega$ *but* (1.2) *does not cover* $A - \lambda\}$. *Then*

$$\operatorname{ess\,sp} A_B = \omega \cup \omega_B.$$

As the proof is technically much more involved than that of Theorem 2.1, only some ingredients will be mentioned: We use the theory of Boutet de Monvel [3] to construct a parametrix for A_B. It makes sense on $L^2(\Omega)$ thanks to the essential observation that $\widetilde{\mathfrak{C}}^{01}\beta^1\widetilde{A}$ is a trace operator *of class* 0 (in the sense of [3]), so acts on $L^2(\Omega)$, in contrast to $\beta^1\widetilde{A}$. Another important point is that ω_B is contained in the *point spectrum* of the symbol of a certain *singular Green operator* of order 0, associated with A_B—examples in [5] show that, in general, $\omega_B \neq \emptyset$.

3.2. For the case where A is strongly elliptic and formally selfadjoint, we find for the selfadjoint, lower bounded realizations (which may be characterized by use of [4]) the analogous results to those in §2.2: ess sp A_B is bounded, and the sequence of eigenvalues going to ∞ behaves approximately like the sequence of eigenvalues for a corresponding selfadjoint realization of P.

REMARK. By similar techniques one may study the spectral theory for the "boundary value problems with potentials" considered in Baouendi-Geymonat [2], and for certain boundary problems for pseudodifferential operators.

BIBLIOGRAPHY

1. S. Agmon, A. Douglis and L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions.* II, Comm. Pure Appl. Math. 17 (1964), 35–92. MR 28 #5252.

2. M. S. Baouendi and G. Geymonat, *Transposition des problèmes aux limites elliptiques*, Symposia Math. 7 (1971), 421–441.

3. L. Boutet de Monvel, *Boundary problems for pseudo-differential operators*, Acta Math. 126 (1971), 11–51.

4. G. Grubb, *Weakly semibounded boundary problems and sesquilinear forms*, Ann. Inst. Fourier (Grenoble) 23 (1973), 145–194.

5. ———, *Essential spectra of elliptic systems on compact manifolds*, Spectral Analysis (C. I. M. E., 1973), Edizioni Cremonese, Roma, 1974, pp. 141–170.

6. B. R. Vaĭnberg and V. V. Grušin, *Uniformly nonelliptic problems.* II, Mat. Sb. 73 (115) (1967), 126–154 = Math. USSR Sb. 2 (1967), 111–134. MR 36 #552.

7. F. Wolf, *On the essential spectrum of partial differential boundary problems*, Comm. Pure Appl. Math. 12 (1959), 221–228. MR 21 #6472.

ISTITUTO MATEMATICO, POLITECNICO DI TORINO, TORINO, ITALY

MATEMATISK INSTITUT, UNIVERSITETSPARKEN 5, 2100 COPENHAGEN, DENMARK