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H-SPACES WITH FINITELY GENERATED COHOMOLOGY ALGEBRAS

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Introduction. The purpose of this note is to announce several results which describe the mod p cohomology ring of an H-space. We assume for the rest of the paper that we are dealing with connected H-spaces with the homotopy type of a CW complex having finitely many cells in each dimension. We use the notation Z_p to denote Z/pZ. Then the cohomology $H^*(X; Z_p)$ is a graded, connected Hopf algebra of finite type, and the cohomology and homology mod p are dual Hopf algebras. If A is a Hopf algebra, $P(A)$ and $Q(A)$ will denote the module of primitives and indecomposables respectively.

There is a secondary cohomology operation which detects the dual of a homology pth power in the mod p cohomology of an H-space. Often, the secondary operation will show that either there is an infinite sequence of nonzero even-dimensional generators of increasing dimension in the cohomology ring, or a given generator is in the image of primary operations occurring in the indeterminacy.

For finite H-spaces, the secondary operation can be used to prove that the third homotopy group has no odd torsion and has two torsion of order at most two. For H-spaces having finitely generated cohomology and no p-torsion of order p, the secondary operation shows that the even generators are concentrated in dimension two for p an odd prime.

A theorem of Milnor and Moore [6] implies that the mod p cohomology of an H-space $X, H^*(X; Z_p)$, is primitively generated if and only if the homology $H_*(X; Z_p)$ is commutative and associative and every element has height less than or equal to p. For H-spaces having $(QH)^{\text{even}}(X; Z_p)$ finite dimensional and $\beta_1(QH)^{\text{even}}(X; Z_p) = 0$, we show that $H^*(X; Z_p)$ is primitively generated if and only if $H_*(X; Z_p)$ is commutative and associative for p odd.

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For H-spaces having primitively generated cohomology $H^*(X; Z_p)$ and no p torsion of order p, we generalize results of Hubbuck [5] which were originally proved using K-theory techniques. Essentially, for odd primes the even generators are in dimension two, and for $p = 2$, the even generators are in dimension $4k + 2$. In either case, the cohomology ring is free commutative.

Finally, given a simply connected finite H-space, there is the standard conjecture that the integral homology of the loop space has no torsion. We have a sufficient condition which ensures that the integral homology of the loop space has no odd torsion. This condition is satisfied by all the simply connected Lie groups except E_8 at the prime three.

This work constitutes part of the author's thesis at Princeton University. I wish to acknowledge the invaluable help of my advisor, John C. Moore. Also I would like to thank Alexander Zabrodsky for many illuminating discussions. Many of the ideas of this work are generalizations of Zabrodsky's Aarhus article [8]. Details and proofs will appear elsewhere.

1. **The main theorem.** Let X be an H-space and let $B(m)$ be the $A(p)$ subalgebra of $H^*(X; Z_p)$ generated by $\Sigma_{j \leq m} H^j(X; Z_p)$. Then $B(m)$ is a subalgebra generated by a coalgebra, so $B(m)$ is a Hopf algebra and

$$Z_p = B(0) \subseteq B(1) \subseteq \cdots \subseteq B(m) \subseteq B(m+1) \subseteq \cdots \subseteq H^*(X; Z_p)$$

filters the cohomology. If $0 \neq \bar{x} \in (QH)^n(X; Z_p)$, there is a representative $x \in H^n(X; Z_p)$ for $\bar{x}$ and an integer m such that $x \in B(m+1), x \notin B(m)$ and $\bar{\Delta}x \in B(m) \otimes B(m)$. The integer m is called the "primitive degree of $\bar{x}$", and x will be called an "m-primitive representative for $\bar{x}$." In general, the primitive degree will be less than or equal to the degree.

The following theorem is the basis for all applications.

MAIN THEOREM 1.1. *Let $\bar{x} \in (QH)^{2n}(X; Z_p)$ be a nonzero indecomposable with primitive degree m, and let x be an m-primitive representative for $\bar{x}$. Suppose $\beta_1 P^n$ factors in $A(p)$: $\beta_1 P^n = \Sigma a_i b_i$, $\deg a_i > 0$, $\deg b_i > 0$. (For $p = 2$, replace β_1 by Sq^1 and P^n by Sq^{2n}.) Suppose further that*

(1) $b_i \bar{x} = 0$ in $(QH)^*(X; Z_p)$;

(2) $\deg b_i \bar{x} \not\equiv 0 \bmod 2p$ for p odd, $\deg b_i \bar{x}$ odd for $p = 2$.

Then x is in the domain of a secondary operation ϕ with $\deg \phi(x) = 2np$. ϕ has indeterminacy in $B(m) + \Sigma \operatorname{im} a_i + D + I(A(p))x$ where D is decomposable and $I(A(p))x$ are elements of the form bx for $b \in I(A(p))$. If

$\bar{\Delta}^{p-1}$ *denotes the iterated* *p-fold reduced diagonal in some fixed order, then*

$$\bar{\Delta}^{p-1}\phi(x) = x \otimes x \otimes \cdots \otimes x + \bar{\Delta}^{p-1}D + \Sigma \operatorname{im} a_i + z \tag{1.1}$$

where

$$z \in \sum_i H^*(X; Z_p) \otimes \cdots \otimes H^*(X; Z_p)I(B(m)) \otimes \cdots \otimes H^*(X; Z_p).$$

REMARK. The a_i act on $H^*(X; Z_p) \otimes \cdots \otimes H^*(X; Z_p)$ via the diagonal action.

2. **Some applications.** In this section we will restrict ourselves to H-spaces X that have the property that $(QH)^{\text{even}}(X; Z_p)$ is finite dimensional and $\beta_1(QH)^{\text{even}}(X; Z_p) = 0$. Browder showed that all H-spaces that have the homotopy type of a finite CW complex satisfy this property [2].

THEOREM 2.1. *If* $n \not\equiv 1 \bmod p$, *then*

$$(QH)^{2n}(X; Z_p) = P^1(QH)^{2n-2(p-1)}(X; Z_p) \quad \textit{for } p \textit{ odd,}$$

and

$$(QH)^{2n}(X; Z_2) = \mathrm{Sq}^2(QH)^{2n-2}(X; Z_2) + \mathrm{Sq}^1(QH)^{2n-1}(X; Z_2) \quad \textit{for } p = 2.$$

OUTLINE OF PROOF. If $n \not\equiv 1 \bmod p$ then

$$\beta_1 P^n = (P^1\beta_1 P^{n-1} - P^n\beta_1)/(n-1) \quad \text{for } p \text{ odd,}$$

$$\mathrm{Sq}^1\mathrm{Sq}^{2n} = \mathrm{Sq}^2\mathrm{Sq}^1\mathrm{Sq}^{2n-2} + \mathrm{Sq}^{2n}\mathrm{Sq}^1 \quad \text{for } p = 2.$$

Suppose there is an $\bar{x} \in (QH)^{2n}(X; Z_p)$ with nonzero projection in

$$(QH)^{2n}(X; Z_p)/\operatorname{im} P^1 \quad \text{for } p \text{ odd,}$$

$$(QH)^{2n}(X; Z_p)/(\operatorname{im} \mathrm{Sq}^2 + \operatorname{im} \mathrm{Sq}^1) \quad \text{for } p = 2.$$

Then there is an m-primitive representative x for $\bar{x}$, and a primitive homology element $t \in (PH)_{2n}(X; Z_p)$ with $\langle x, t\rangle \neq 0$ and

$$\langle \operatorname{im} P^1 + B(m), t\rangle = 0 \quad \text{for } p \text{ odd,}$$

$$\langle \operatorname{im} \mathrm{Sq}^2 + \operatorname{im} \mathrm{Sq}^1 + B(m), t\rangle = 0 \quad \text{for } p = 2.$$

This implies

$$\langle \text{im } P^1 + B(m), t^p \rangle = 0 \quad \text{for } p \text{ odd},$$

$$\langle \text{im Sq}^2 + \text{im Sq}^1 + B(m), t^2 \rangle = 0 \quad \text{for } p = 2.$$

By the Main Theorem and the assumptions of this section, x is in the domain of a secondary operation ϕ with

$$b_1 = \beta_1 P^{n-1} \qquad b_2 = \beta_1 \quad \text{for } p \text{ odd},$$

$$b_1 = \text{Sq}^1\text{Sq}^{2n-2}, \qquad b_2 = \text{Sq}^1 \quad \text{for } p = 2.$$

By formula (1.1), $\langle \phi(x), t^p \rangle \neq 0$. Further, there is a representative x_1 for $\phi(x)$ and x_1 has nonzero projection in

$$(QH)^{2np}(X; Z_p)/\text{im } P^1 \qquad \text{for } p \text{ odd},$$

$$(QH)^{4n}(X; Z_2)/(\text{im Sq}^2 + \text{im Sq}^1) \quad \text{for } p = 2.$$

Since $np \not\equiv 1 \bmod p$, this process is repeatable. By induction, there exists an infinite sequence of nonzero even-dimensional generators. This contradicts the assumption that $(QH)^{\text{even}}(X; Z_p)$ is finite dimensional. This completes the proof. Q.E.D.

Theorem 2.1 implies $(QH)^{2np}(X; Z_p) = P^1(QH)^{2np-2(p-1)}(X; Z_p)$ for p odd, and $(QH)^4(X; Z_p) = 0$ for p odd, $(QH)^4(X; Z_2) = \text{Sq}^1(QH)^3(X; Z_2)$. Using this data, we can prove the following corollaries:

COROLLARY 2.2. *Let p be an odd prime. If $H_*(X; Z_p)$ is commutative and associative as a ring, then $H^*(X; Z_p)$ is primitively generated.*

PROOF. Essentially there can be no pth powers in homology because $\langle \text{im } P^1, t^p \rangle = 0$. Q.E.D.

COROLLARY 2.3. *Let X be a simply connected H-space having the homotopy type of a finite complex. Then the third homotopy group has no odd torsion and has two torsion of order at most two.*

PROOF. By the universal coefficient theorem and the Hurewicz theorem, and the fact that X is two connected [2],

$$(QH)^4(X; Z_p) \approx H^4(X; Z_p) \approx \text{torsion } \pi_3(X) \otimes Z_p.$$

Therefore $\pi_3(X)$ has no odd torsion, and by the Bockstein spectral sequence, $\pi_3(X)$ has two torsion of order at most two. Q.E.D.

Corollary 2.2 is related to work of Browder [1]. Corollary 2.3 resolves a conjecture of Clark [3].

Given a simply connected finite H-space X, Browder, Clark, Gitler and others tried to prove that $H_*(\Omega X; Z)$ is torsion free. I have the following sufficient condition for odd primes:

THEOREM 2.4. *Let p be an odd prime and $v(j) = 1 + p + \cdots + p^j$, for $j = 1, 2, \cdots$. Let X be a simply connected H-space having the homotopy type of a finite complex. Then if $P^{p^j}(QH)^{2v(j)}(X; Z_p) = 0$ for all j, $H_*(\Omega X; Z)$ has no p torsion.*

The hypotheses of Theorem 2.4 are satisfied for all simply connected Lie groups except E_8 mod 3.

3. ***H*-spaces having no p torsion of order p.** In a series of papers, Hubbuck studied torsion free H-spaces using K-theory techniques [4], [5]. In this last section, we indicate that his results may be generalized to the class of H-spaces having no p torsion of order p.

THEOREM 3.1. *Let p be an odd prime and let X be an H-space with $\beta_1 H^*(X; Z_p) = 0$ and $(QH)^{even}(X; Z_p)$ finite dimensional. Then the even-dimensional generators are all in dimension two.*

THEOREM 3.2. *Let X be an H-space with $H^*(X; Z_p)$ primitively generated and $\beta_1 H^*(X; Z_p) = 0$. Then if p is an odd prime, $H^*(X; Z_p) \approx Z_p[2] \otimes E$, where $Z_p[2]$ is a free polynomial algebra on primitive generators of dimension two and E is an exterior algebra on odd-dimensional generators.*

THEOREM 3.3. *Let X be a two-connected H-space with $H^*(X; Z_2)$ primitively generated and $\mathrm{Sq}^1 H^*(X; Z_2) = 0$. Then $H^*(X; Z_2) \approx Z_2[4k + 2] \otimes E$, where $Z_2[4k + 2]$ is a free polynomial algebra on primitive generators of dimension $4k + 2$, and E is an exterior algebra on odd-dimensional generators. Further,*

$$\mathrm{Sq}^{4k}: (QH)^{4k+2}(X; Z_p) \longrightarrow (QH)^{8k+2}(X; Z_p)$$

is a monomorphism for all k.

In Theorems 3.2 and 3.3 we make no assumption about $(QH)^{\mathrm{even}}(X; Z_p)$.

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