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SOLVABILITY ON MANIFOLDS BY QUADRATURES PERMITTING ONLY INTEGRALS

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Let M be a connected C^∞ manifold, and let $\rho\colon \widetilde{M} \longrightarrow M$ be the universal covering map. Choose a base point $\widetilde{x}_0 \in \widetilde{M}$, and write $x_0 = \rho\widetilde{x}_0$. The fundamental group $\pi_1(M)$ is assumed to be finitely generated.

Let A be a subcomplex of the de Rham complex $\Lambda(M)$ satisfying the conditions:

(a) The subcomplex A is closed under the exterior product.

(b) The inclusion $A \subset \Lambda(M)$ induces an isomorphism $H(A) \approx H(\Lambda(M))$. Write $F_0 = \rho^*A^0$ and $\Omega = \rho^*A^1$. If $w \in A^1$ is a closed 1-form on M, then the integral $\int_{\widetilde{x}_0} \rho^*w$ is a function on $\widetilde{M}$ and can be regarded as a multivalued function on M. All such integrals together with 1 span a vector space F_1 of functions on $\widetilde{M}$ such that $F_0 \subset F_1$. For $r \geqslant 1$, define F_{r+1} to be the vector space of functions spanned by F_r and all $\int_{\widetilde{x}_0} w$, w being closed 1-forms belonging to the subspace $F_r\Omega$ of $\Lambda^1(\widetilde{M})$. It turns out that $\widetilde{F} = \bigcup_{r\geqslant 0} F_r$ is an algebra of functions on $\widetilde{M}$.

Recall that the lower central series of a group G consists of commutator subgroups G_r, $r \geqslant 1$, defined by $G_1 = G$ and $G_{r+1} = [G_r, G]$, $r \geqslant 1$. The lower central series is said to stabilize modulo torsion if G_r/G_{r+1} is finite for r sufficiently large. A group G is said to be torsion free residually nilpotent if each quotient G_r/G_{r+1} is torsion free and if $\bigcap G_r = \{e\}$.

The purpose of this note is to announce the next results, which will be proved in detail elsewhere.

THEOREM 1. *The algebra $\widetilde{F}$ is finitely generated over F if and only*

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if the lower central series of $\pi_1(M)$ *stabilizes modulo torsion.*

THEOREM 2. *The algebra* $\widetilde{F}$ *of functions on* $\widetilde{M}$ *separates* $\widetilde{M}$ *if and only if* $\pi_1(M)$ *is torsion free residually nilpotent.*

COROLLARY 1. *The algebra* $\widetilde{F}$ *is finitely generated and separates* $\widetilde{M}$ *if and only if* $\pi_1(M)$ *is torsion free nilpotent.*

COROLLARY 2. *If* M *is a compact Riemann surface, then* $\widetilde{F}$ *separates the universal covering surface* $\widetilde{M}$.

The function algebra $\widetilde{F}$ is obtained from the given function algebra F on M by adjoining multivalued functions which are obtained through iterated integration. According to the above theorems, we know precisely when $\widetilde{F}$ can be obtained from F by adjoining a finite number of elements and also when every continuous function on $\widetilde{M}$ can be approximated on compact sets by functions in $\widetilde{F}$. Thus our results provide answers to questions pertaining to a several independent variable version of the Picard-Vessiot theory. In the one variable case, it is known [6] that an extension of a differential field by integrals corresponds to a Galois group which is algebraic nilpotent.

Since a compact nilmanifold has a torsion free nilpotent fundamental group, this work also relates to the function theory on nilpotent Lie groups under a discrete subgroup action such as the continuous theta function theory by Auslander and Rauch [1].

In order to prove Theorems 1 and 2, observe that $\widetilde{F}$ can be regarded as an algebra of iterated integrals of 1-forms on M, whose value along each path depends only on the path homotopy class. By restricting to the space of loops at x_0, we obtain from $\widetilde{F}$ a quotient algebra $\widetilde{F}'$ which has an ascending filtration. We may take $\widetilde{F}'$ as an algebra of functions on $\pi_1(M)$.

Theorem 1 is equivalent to a necessary and sufficient condition for $\widetilde{F}'$ being finitely generated over the real (or complex) number field, and Theorem 2 reduces to a necessary and sufficient condition for $\widetilde{F}'$ to separate $\pi_1(M)$. It remains to show that $\widetilde{F}' = F'_A$, where F'_A is defined as in [3]. The inclusion $\widetilde{F}' \subset F'_A$ is not difficult to see. Using a method of formal power series connections as described in [5], we are able to establish $F'_A \subset \widetilde{F}'$.

Corollary 2 follows from Theorem 2 because of a result of Baumslag [2] which implies that $\pi_1(M)$ is torsion free residually nilpotent.

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