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A CONJECTURE OF M. GOLOMB ON OPTIMAL AND NEARLY-OPTIMAL LINEAR APPROXIMATION

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In 1964, M. Golomb, in his survey paper on optimal and nearly-optimal linear approximation, presented at the General Motors Conference [3], called attention to an unsolved problem. It is the purpose of this note to solve this problem and at the same time to give a certain extension of the Haršiladze-Lozinskiĭ theorem.

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Let $C_{2\pi}$ be the space of continuous 2π-periodic functions with Čebyšev norm, Π_n the class of trigonometric polynomials of degree $\leqslant n$, and $E_n[f] = \inf\{\|f-p\|; p \in \Pi_n\}$ the error of best approximation of an $f \in C_{2\pi}$ by elements of Π_n for an $n \in \mathbf{P} = \{0, 1, 2, \cdots\}$. A sequence $\{U_n\}_{n \in \mathbf{P}}$ of bounded linear operators on $C_{2\pi}$ into $C_{2\pi}$ is called *asymptotically optimal* [3] for a given subset $Y \subset C_{2\pi}$ if

$$\sup_{f \in Y} \|f - U_n f\| \leqslant M_Y \sup_{f \in Y} E_n[f] \qquad (n \in \mathbf{P}), \tag{1}$$

M_Y being some constant. $\{U_n\}$ is called *optimal* for Y if (1) is satisfied with $M_Y = 1$.

In particular, Y will be taken to be one of the spaces C_0^r, $r \in \mathbf{P}$ or A_0^α, $\alpha > 0$, where C_0^r consists of those $f \in C_{2\pi}$ whose rth derivative is continuous and satisfies $\|f^{(r)}\| \leqslant 1$, and A_0^α is the class of functions $f(z)$ of a complex variable $z = x + iy$ which are 2π-periodic in x, real for $y = 0$, analytic in the open strip $|y| < \alpha$, continuous in $|y| \leqslant \alpha$, and satisfy

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$\sup_{|y| \leqslant \alpha, |x| \leqslant \pi} |\mathrm{Re}\, f(z)| \leqslant 1$. By the Jackson and Bernstein theorems, a sequence $\{U_n\}$ of bounded linear operators is asymptotically optimal for come C_0^r [some A_0^α] iff $\|f - U_n f\| = O(n^{-r})$ $[O(e^{-\alpha n})]$, $n \longrightarrow \infty$, for all $f \in C_0^r$ $[f \in A_0^\alpha]$. Moreover, since $\sup\{E_n[f]; f \in C_0^r\} = \mu_r (n+1)^{-r}$ for all $n \in \mathbf{P}$, where μ_r denote the Favard-Achieser-Kreĭn constants $(r \in \mathbf{P})$, a sequence $\{U_n\}$ is optimal for some C_0^r iff $\|f - U_n f\| \leqslant \mu_r (n+1)^{-r}$ for all $f \in C_0^r$, $n \in \mathbf{P}$.

Golomb's conjecture [3] consists of the following two statements.

(A) *There does not exist a sequence* $\{U_n\}$ *of bounded linear polynomial (i.e.* $U_n(C_{2\pi}) \subset \Pi_n$ *for all* $n \in \mathbf{P}$) *operators which is asymptotically optimal for all the classes* C_0^r, $r \in \mathbf{P}$, *and at the same time for all the classes* A_0^α, $\alpha > 0$.

(B) *There does not exist a sequence of bounded linear polynomial operators which is optimal for all classes* C_0^r, $r \in \mathbf{P}$.

In case (A), this was motivated by the fact that the Fourier partial sums S_n are asymptotically optimal for each A_0^α, $\alpha > 0$, but not for any C_0^r, $r \in \mathbf{P}$, whereas the de La Vallée Poussin sums $V_n = (n - [n/2] + 1)^{-1} \cdot \Sigma_{k=[n/2]}^n S_k$ are asymptotically optimal for each C_0^r, $r \in \mathbf{P}$, but not for any A_0^α, $\alpha > 0$. Concerning (B), for each class C_0^r there exists an optimal sequence of convolution type operators, but it depends on r and is unique at least among convolutions.

To prove (A) assume the contrary to be valid. If $\{U_n\}$ is the sequence in question, define a sequence $\{\overline{U}_n\}$ of bounded linear polynomial operators by

$$\overline{U}_n f = \frac{1}{2\pi} \int_{-\pi}^{\pi} T_{-t} U_n T_t f\, dt, \qquad T_t f(x) = f(x+t), \tag{2}$$

according to Marcinkiewicz' device [5]. Then the $\overline{U}_n$ are convolutions and they are asymptotically optimal for all C_0^r, A_0^α, $r \in \mathbf{P}, \alpha > 0$ as well. Thus the following two theorems may be applied in order to derive a contradiction.

THEOREM 1. *If* $\{U_n\}$ *is a sequence of bounded linear polynomial operators on* $C_{2\pi}$ *which is asymptotically optimal for some* A_0^α, $\alpha > 0$, *then* $\limsup_{n\to\infty} \|U_n\| = +\infty$.

THEOREM 2. *If* $\{U_n\}$ *is a sequence of bounded linear polynomial convolution operators on* $C_{2\pi}$ *which is asymptotically optimal for some* C_0^r, $r \in \mathbf{P}$, *then* $\|U_n\| = O(1)$, $n \longrightarrow \infty$.

The proof of Theorem 1 proceeds via (2) and makes use of a weak version of an inequality of Hardy-Littlewood [4] and Sidon [8] (to be found e.g. in Nikol'skii [6, p. 262]). Theorem 2 is proved by an application of Bernstein's inequality to $(U_n - V_n)f$.

For the proof of (B) assume that $\{U_n\}$ satisfies $\|f - U_n f\| \leqslant \mu_r(n+1)^{-r}$ for all $f \in C_0^r$, n, $r \in \mathbf{P}$. Then the following Lemma furnishes a contradiction to the fact that the μ_r are bounded uniformly in r.

LEMMA. *If $\{U_n\}$ is a sequence of bounded linear polynomial operators on $C_{2\pi}$ such that for each $r \in \mathbf{P}$*

$$\sup_{f \in C_0^r} \|f - U_n f\| \leqslant M_r(n+1)^{-r} \qquad (f \in C_0^r, n \in \mathbf{P}), \tag{3}$$

then $\limsup_{r \to \infty} M_r = +\infty$.

This is a consequence of (2) and of the inequality mentioned above (see [8]).

In this context let us mention the familiar Haršiladze-Lozinskiĭ theorem (see e.g. [2, pp. 212, 233]) which asserts that there does not exist a sequence $\{U_n\}$ of bounded linear polynomial operators satisfying simultaneously

(a) $U_n(U_n f) = U_n f$ for each $n \in \mathbf{P}, f \in C_{2\pi}$, and

(b) $\|f - U_n f\| \longrightarrow 0$ as $n \longrightarrow \infty$ for each $f \in C_{2\pi}$.

Extensions of this result have been given e.g. by Berman [1] and Sapogov [7]. As a consequence of the above, another extension is obtained on replacing the projection condition (a) by (a′) or (a″) below.

(a′) $\{U_n\}$ is asymptotically optimal for some A_0^α, $\alpha > 0$.

(a″) $\{U_n\}$ satisfies (3) for each $r \in \mathbf{P}$, and $M_r = O(1)$, $r \longrightarrow \infty$.

Details will appear elsewhere.

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