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ACCESSIBILITY AND FOLIATIONS WITH SINGULARITIES

BY P. STEFAN

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Introduction. Recently, Sussmann has proved that the accessible sets of a system of vectorfields on a C^∞ manifold M are immersed submanifolds of M [**1**], [**2**]. In this paper we state a general theorem on accessible sets of collections of 'arrows' and indicate how it implies (a) the above result; (b) the fact that the orbits of an arbitrary 'isotopically connected' subgroup of Diff(M) form a foliation with singularities; and (c) a similar result for groupoids of germs of local diffeomorphisms. The complete proofs will be published elsewhere.

The results of this paper were obtained independently of Sussmann's work.

1. **Statement of the main result.** Throughout this paper, the word 'differentiable' refers to a fixed class C^q, $1 \leqq q \leqq \infty$, and M is a finite-dimensional paracompact differentiable manifold. Theorems 1 and 4 are also valid in the real analytic case.

A subset L of M is said to be a *k-leaf* of M if there exists a differentiable structure σ on L such that (i) (L, σ) is a connected k-dimensional immersed submanifold of M, and (ii) if N is an arbitrary locally connected topological space, and $f: N \to M$ is a continuous function such that $f(N) \subset L$, then $f: N \to (L, \sigma)$ is continuous.

It follows from the properties of immersions that if $f: N \to M$ is a differentiable mapping of manifolds such that $f(N) \subset L$, then $f: N \to (L, \sigma)$ is also differentiable. In particular, σ is the unique differentiable structure on L which makes L into an immersed k-dimensional submanifold of M. Since M is paracompact, every connected immersed submanifold of M is separable, and so the dimension k of a leaf L is uniquely determined.

We say that F is a *C^q-foliation of M with singularities* if F is a partition of M into C^q-leaves of M such that, for every $x \in M$, there exists a local C^q-chart ψ of M with the following properties:

(a) The domain of ψ is of the form $U \times W$, where U is an open neighbourhood of 0 in R^k, W is an open neighbourhood of 0 in R^{n-k}, and k is the dimension of the leaf through x.

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(b) $\psi(0, 0)=x$.

(c) If L is a leaf of $\boldsymbol{F}$, then $L\cap\psi(U\times W)=\psi(U\times l)$, where $l=\{w \in W: \psi(0, w) \in L\}$.

We write $(M, \boldsymbol{F})$ for the C^q-manifold with the same underlying set as M and with the C^q-structure of the disjoint sum of leaves of $\boldsymbol{F}$.

By a *local diffeomorphism* of M we mean a diffeomorphism of one open subset of M onto another. We say that a differentiable function $a: R\times M\to M$ is an *arrow* if its domain is an open subset of $R\times M$ and if it satisfies the following two conditions: (i) for every $t \in R$, $a^t=a(t, -)$ is a local diffeomorphism of M (possibly with the empty domain), and (ii) if (t, x) belongs to the domain of a, then so does (s, x) for every s between 0 and t and $a(0, x)=x$. We write $\dot{a}(t, x)$ for the tangent vector at t of the curve $a(-, x)$, and $(a^t)^*(x)$ for the differential at x of the function $a^t: M\to M$. If $y=a(t, x)$ then $\dot{a}(t, x) \in T_yM$, and $(a^t)^*(x)$ is a linear mapping $T_xM\to T_yM$.

If A is a set of arrows on M, we let ΨA denote the collection of all the local diffeomorphisms ϕ of the form

$$\phi = a_1^{t_1} \circ a_2^{t_2} \circ \cdots \circ a_p^{t_p}$$

for some $a_i \in A$ and $t_i \in R$. $A(x)$ and $\bar{A}(x)$ denote the vector subspaces of T_xM spanned respectively by the set $\{\dot{a}(t, y): a \in A,\ a(t, y)=x\}$ and by $\bigcup \phi^*(y)\cdot A(y)$, where the union extends over all ϕ and y such that $\phi \in \Psi A$ and $\phi(y)=x$.

We say that A is *symmetric* if $\phi \in \Psi A$ implies $\phi^{-1} \in \Psi A$ and that it is *homogeneous* if $(a^t)^*(x)\cdot A(x)\subset A(y)$ whenever $a \in A$ and $a(t, x)=y$.

We write $y=x \bmod A$ if $y=x$ or if $y=\phi(x)$ for some $\phi \in \Psi A$. If A is symmetric, then this is an equivalence relation on M; its equivalence classes are termed here the *accessible sets* of A.

Let $\sim$ be an equivalence relation on M. We say that an arrow a (or a local diffeomorphism ϕ) *respects* $\sim$ if $a(t, x)\sim x$ whenever (t, x) belongs to the domain of a (or $\phi(x)\sim\phi(y)$ whenever $x\sim y$ and both x and y belong to the domain of ϕ).

PROPOSITION. *Let $\sim$ be an equivalence relation on M and let $\tilde{A}$ be the class of all arrows on M which respect $\sim$. Then $\tilde{A}$ is symmetric and homogeneous.*

COROLLARY. *Every symmetric set of arrows is contained in a homogeneous symmetric set of arrows whose accessible sets are the same.*

PROOF OF THE COROLLARY. Take $\sim$ to be the relation $x=y$ (mod A).

THEOREM 1. (a) *Let A be a symmetric set of arrows on M and let $\boldsymbol{F}=\boldsymbol{F}(A)$ be the partition of M into the accessible sets of A. Then $\boldsymbol{F}$ is a* *foliation with singularities. In particular, every accessible set of A is a leaf of M, and thus admits a unique differentiable structure of a connected immersed submanifold of M.*

(b) $T_x(M, \boldsymbol{F})=\bar{A}(x)$ *for every* $x \in M$. *In particular,* $T_x(M, \boldsymbol{F})=A(x)$ *for every* $x \in M$ *if and only if A is a homogeneous set of arrows.*

(c) *Let* $\sim$ *be an equivalence relation on M and let* $A=\tilde{A}$. *If* $\phi \in$ Loc Diff(M) *and* ϕ *respects* $\sim$, *then* $\phi \in$ Loc Diff(M, $\boldsymbol{F}$).

2. **Foliations generated by subgroups of** Diff(M). Let G be a subgroup of Diff(M). Two elements g and h of G are said to be G-isotopic if there exists a differentiable mapping $a: R \times M \to M$ such that $a^t \in G$ for every $t \in R$, $a^t=g$ for $t \leqq 0$, and $a^t=h$ for $t \geqq 1$. The isotopy component G_0 of the identity is a normal subgroup of G.

THEOREM 2. (a) *Let* $\boldsymbol{F}=\boldsymbol{F}(G_0)$ *be the partition of M into G_0-orbits. Then $\boldsymbol{F}$ is a foliation with singularities.*

(b) $G \subset$ Diff(M, $\boldsymbol{F}$) *and every G-orbit consists of G_0-orbits of the same dimension.*

(c) *If* G/G_0 *is countable, then every G-orbit admits a unique structure of a separable immersed submanifold of M.*

PROOF. Let A be the set of all differentiable mappings $R \times M \to M$ such that $a^t \in G$ for every $t \in R$, and $a^t=\mathrm{id}_M$ for $t \leqq 0$. Then A is a symmetric set of arrows and the accessible sets of A are the orbits of G_0.

3. **Foliations generated by groupoids.** If $\phi \in$ Loc Diff(M) and x belongs to the domain of ϕ, let $\gamma(x, \phi)$ denote the germ of ϕ at x. Let $e(x)=\gamma(x, \mathrm{id}_M)$, and let $\Delta=\Delta(M)$ be the groupoid of all germs of local diffeomorphisms of M. Let $\alpha: \Delta \to M$ be the projection onto the initial point, so that $\alpha(\gamma(x, \phi))=x$.

Let Γ be a subgroupoid of Δ. We say that $g \in \Gamma$ and $h \in \Gamma$ are Γ-isotopic if $\alpha(g)=\alpha(h)$ and if there exists an open neighborhood U of $\alpha(g)$ and a differentiable mapping $a: R \times U \to M$ such that (i) $\gamma(\alpha(g), a^t) \in \Gamma$ for every $t \in R$, (ii) $\gamma(\alpha(g), a^t)=g$ for $t \leqq 0$, and (iii) $\gamma(\alpha(g), a^t)=h$ for $t \geqq 1$. Let $\Gamma_0=\{g \in \Gamma : g$ is Γ-isotopic to $e(\alpha(g))\}$. Then Γ_0 is a subgroupoid of Γ and g and h are Γ-isotopic if and only if $\Gamma_0 g=\Gamma_0 h$.

THEOREM 3. *The assertions of Theorem* 2 *remain valid if we replace G by* Γ, G_0 *by* Γ_0 *and* Diff(M, $\boldsymbol{F}$) *by* $\Delta(M, \boldsymbol{F})$.

4. **Foliations generated by systems of vectorfields.** If X is a differentiable vectorfield on M, exp X denotes the *flow* of X, so that $t \to \exp X.(t, x)$ is the integral curve of X passing through x at $t=0$. If S is a set of vectorfields on M, we write $S(x)$ for the subspace of $T_x M$ spanned by $\{X(x): X \in S\}$, and put $\exp S=\{\exp X : X \in S\}$. It is clear that exp S is a symmetric

set of arrows; the *accessible sets* of S are, by definition, the accessible sets of $\exp S$.

THEOREM 4. (a) *Let $F=F(S)$ be a partition of M into the accessible sets of S. Then F is a foliation with singularities.*

(b) *$T_x(M, F)=S(x)$ for every $x \in M$ if and only if* $(\exp X^t)^*(x).S(x) \subset S(y)$ *whenever $X \in S$ and* $\exp X.(t, x)=y$.

COROLLARY. *Let $\sim$ be an arbitrary equivalence relation on M, and let S be the set of all vectorfields on M that leave the equivalence classes of $\sim$ invariant. Then S is closed under formation of the Lie bracket.*

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SCHOOL OF MATHEMATICS AND COMPUTER SCIENCE, UNIVERSITY COLLEGE OF NORTH WALES, BANGOR LL57 2UW, UNITED KINGDOM