BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 80, Number 6, November 1974

L^p-CONVOLUTION OPERATORS AND TENSOR PRODUCTS OF BANACH SPACES

BY JOHN E. GILBERT[1]

Communicated by Kenneth A. Ross, March 18, 1974

Associated with every locally compact group are triples of Banach algebras

$$\{\mathscr{C}_0(G), L^1(G), L^\infty(G)\}, \qquad \{A(G), C^*(G), B(G)\} \tag{1}$$

intimately connected with duality theory (the notation is that of **[1]**). In both cases the middle algebra is the closure of $L^1(G)$ in the dual of the first algebra and also the predual of the third algebra (at least when G is amenable in the second case). Furthermore, the third algebra is closely connected with the multiplier algebra of the first algebra.

For abelian groups, compact or discrete, Varopoulos **[11]**, **[12]** showed to great effect how the second triple could be obtained and studied by starting with the tensor product $\mathscr{C}_0(G) \otimes_\gamma \mathscr{C}_0(G)$, γ the greatest cross-norm. An analogous construction starting this time with $\mathscr{C}_0(G) \otimes_\lambda \mathscr{C}_0(G)$, λ the least cross-norm, would produce the first triple. On the other hand, at least for amenable groups, the triples in (1) can be considered as the extreme case $p=1, 2$, respectively, of a family $\{A^p(G), cv^p(G), B^p(G)\}$, $1 \leqq p \leqq 2$, associated with L^p-convolution operator theory, and obtained by starting with the tensor product $L^{p'}(G) \otimes_\gamma L^p(G), p \neq 1$, or $\mathscr{C}_0(G) \otimes_\gamma L^1(G)$, $p=1$. Indeed, Herz has shown that $A^p(G)$ is a pointwise Banach algebra **[6]** while $B^p(G)$, $1<p\leqq 2$, is both the multiplier algebra of $A^p(G)$ and the Banach dual space of $cv^p(G)$, G amenable. In these notes we outline a new approach to convolution operator theory, by starting with $\mathscr{C}_0(G) \otimes_\alpha \mathscr{C}_0(G)$, α a tensorial norm **[5]**, rather than with $L^{p'}(G) \otimes_\gamma L^p(G)$. A triple $\{\mathscr{V}^\alpha(G), \mathscr{L}^{\alpha'}(G), \mathscr{W}^\alpha(G)\}$ analogous to (1) is obtained. For L^p-convolution operator theory, a family of tensorial norms α_{pq} is used. The two basic ideas are to exploit classical Banach space theory concerning $L^p(\mu)$-spaces, for example, forgetting about group structure, and then, when a group structure is imposed, to exploit standard $\mathscr{C}_0(G)$- and $L^1(G)$-techniques because all the 'L^p-theory' has been thrown into the norm α_{pq},

AMS (*MOS*) *subject classifications* (1970). Primary 43A22, 46J10; Secondary 43A15, 43A20, 47D15, 46B10.

[1] Supported in part by NSF Grant No. GP-38865.

associated with which is a highly developed operator ideal theory (cf. [2], [4]). Solutions to a number of open problems are obtained (cf. §4). Detailed proofs will appear elsewhere.

1. **Varopoulos spaces, algebras.** By a *Varopoulos space* we shall mean any Banach space $V^\alpha(X, Y)$ of the form $V^\alpha(X, Y)=\mathscr{C}_0(X)\otimes_\alpha \mathscr{C}_0(Y)$, where X, Y are locally compact Hausdorff spaces and α a tensorial norm. The simplest such space is $V^\lambda(X, Y)=\mathscr{C}_0(X\times Y)$, and there is always a norm-decreasing isomorphism $\sum_\alpha^\lambda: V^\alpha(X, Y)\to V^\lambda(X, Y)$. Thus

$$\mathscr{C}_0(X)\otimes_\gamma \mathscr{C}_0(Y) = V^\gamma(X, Y) \subseteq V^\alpha(X, Y) \subseteq V^\lambda(X, Y) = \mathscr{C}_0(X \times Y).$$

A Varopoulos space $V^\alpha(X, Y)$ is said to be a *Varopoulos algebra* if $\alpha(f\cdot g)\leqq \alpha(f)\alpha(g)$ for the pointwise product of $f, g\in\mathscr{C}_0(X)\otimes\mathscr{C}_0(Y)$ ($\alpha(\cdot)$ norm on $V^\alpha(X, Y)$).

THEOREM 1. *Each Varopoulos algebra $V^\alpha(X, Y)$ is a commutative semisimple Banach algebra with maximal ideal space $X\times Y$. Furthermore, $V^\alpha(X, Y)$ is regular and selfadjoint.*

Both $V^\lambda(X, Y)$ and $V^\gamma(X, Y)$ are Varopoulos algebras, and any $V^\alpha(X, Y)$ is a Banach $V^\gamma(X, Y)$-module. More generally, for any sequence $\{x_n\}$ (finite or infinite) in any Banach space $\mathfrak{X}$, set

$$M_r(\{x_n\}) = \sup\left\{\left(\sum_n |\langle x^*, x_n\rangle|^r\right)^{1/r} : x^*\in\mathfrak{X}^*, \|x^*\|\leqq 1\right\}, \qquad r\neq\infty,$$

($M_\infty(\{x_n\})=\sup_n \|x_n\|$). When $1\leqq q\leqq p\leqq\infty$ and $t\in\mathfrak{X}\otimes\mathfrak{Y}$ set

$$\alpha_{pq}(t) = \alpha_{pq}(t; \mathfrak{X}, \mathfrak{Y}) = \inf\left(\sum_n |\lambda_n|^r\right)^{1/r} M_p(\{x_n\})M_{q'}(\{y_n\}),$$

where $1/q+1/q'=1$, $1/p+1/q'+1/r=1$ and the infimum is taken over all representations $t=\sum_n \lambda_n x_n\otimes y_n$ (cf. [8]). If $\mathfrak{X}=\mathscr{C}_0(X)$, then

$$M_r(\{f_n\}) = \sup\left\{\left(\sum_n |f_n(x)|^r\right)^{1/r} : x\in X\right\},$$

and so we deduce

THEOREM 2. *The Varopoulos space $V^{pq}(X)$, $1\leqq p\leqq q\leqq\infty$, obtained by taking $\alpha=\alpha_{p'q'}$ is always a Varopoulos algebra.*

2. **Fundamental properties.** Deep Banach space results yield properties of $V^{pq}(X, Y)$. For any pair X, Y

$$\mathscr{C}_0(X)\otimes_\gamma \mathscr{C}_0(Y) = V^{1\infty}(X, Y) \subseteq V^{pq}(X, Y) \subseteq V^{11}(X, Y) = \mathscr{C}_0(X \times Y)$$

isometrically or with norm-decreasing inclusion. The right-hand equality follows from the fact that $\mathscr{C}_0(Y)$ is an $\mathscr{L}^{\infty}_{1+\varepsilon}$-space for every $\varepsilon>0$ [9]. From the Grothendieck 'fundamental theorem for metric spaces' [5], [9] we obtain

THEOREM 3. *Up to equivalence of norms, $V^{22}(X, Y)=\mathscr{C}_0(X)\otimes_\gamma \mathscr{C}_0(Y)$; in fact,*

$$\alpha_{22}(f) \leqq \gamma(f) \leqq K_G\alpha_{22}(f), \qquad f\in\mathscr{C}_0(X)\otimes\mathscr{C}_0(Y),$$

where K_G is the Grothendieck universal constant.

THEOREM 4 (KWAPIEN-PIETSCH). *A linear operator $T:\mathscr{C}_0(Y)\to M(X)$ belongs to $(V^{pq}(X, Y))^*$ if and only if for each $\varepsilon>0$ there exist probability measures μ on X and ν on Y such that*

$$|\langle f, Tg\rangle| \leqq (1+\varepsilon)\,\|T\|_{pq}\left(\int_X |f|^{p'}\,d\mu\right)^{1/p'}\left(\int_Y |g|^q\,d\nu\right)^{1/q}$$

for all $f\in\mathscr{C}_0(X)$, $g\in\mathscr{C}_0(Y)$.

Use of the bilinear Riesz-Thorin theorem and Theorem 3 now gives

THEOREM 5. (i) *If $1\leqq p\leqq 2$ and $2\leqq q\leqq\infty$, then $V^{pq}(X, Y)=V^\gamma(X, Y)$ up to equivalence of norms.*

(ii) *If $1\leqq p\leqq q\leqq 2$ or $2\leqq p\leqq q\leqq\infty$, then $V^{rs}(X, Y)\subseteq V^{pq}(X, Y)$ where*

$$\frac{1}{r}=\frac{1-\theta}{2}+\frac{\theta}{p}, \qquad \frac{1}{s}=\frac{1-\theta}{2}+\frac{\theta}{q}, \qquad 0<\theta<1,$$

the embedding being continuous.

(iii) *If $1\leqq p<q<2$, then $V^{pq}(X, Y)=V^{1q}(X, Y)$ up to equivalence of norms.*

The proof of property (iii) in Theorem 5 also uses the fact that a bounded linear operator $T:\mathscr{C}(S)\to L^r(\mu)$ automatically is absolutely s-summing if $2<r<s\leqq\infty$ (cf. [10]).

3. **The triple $\{\mathscr{V}^\alpha(G), \mathscr{L}^{\alpha'}(G), \mathscr{W}^\alpha(G)\}$.** Let G be a locally compact group, and $L^1(G)\otimes_{\alpha'} L^1(G)$ the completion of $L^1(G)\otimes L^1(G)$ with respect to the associate norm α' of α [5]; equivalently, $L^1(G)\otimes_{\alpha'} L^1(G)$ is the closure of $L^1(G\times G)$ in $(V^\alpha(G, G))^*$. Starting from $\mathscr{C}_0(G)\otimes_\alpha\mathscr{C}_0(G)$, $L^1(G)\otimes_{\alpha'} L^1(G)$, $(L^1(G)\otimes_{\alpha'} L^1(G))^*$, we define a triple $\{\mathscr{V}^\alpha(G), \mathscr{L}^{\alpha'}(G), \mathscr{W}^\alpha(G)\}$. Now from Theorem 3 it follows (nontrivially!) that $(L^1(G)\otimes_{\alpha'} L^1(G))^*$ always contains $M(A(G))$ where $(M\phi)(x, y)=\phi(xy^{-1})$.

DEFINITION 1. $\mathscr{V}^\alpha(G)$ will denote the completion of $A(G)$ with respect to the norm induced on $M(A(G))$ by $(L^1(G)\otimes_{\alpha'} L^1(G))^*$.

Clearly

$$A(G)\subseteq\mathscr{V}^\gamma(G)\subseteq\mathscr{V}^\alpha(G)\subseteq\mathscr{V}^\lambda(G)=\mathscr{C}_0(G).$$

The closure of $L^1(G)$ in $(\mathscr{V}^\alpha(G))^*$ is denoted by $\mathscr{L}^{\alpha'}(G)$. Then

$$L^1(G) = \mathscr{L}^\gamma(G) \subseteq \mathscr{L}^{\alpha'}(G) \subseteq C^*(G).$$

Finally, set $\mathscr{W}^\alpha(G)=\{\phi \in L^\infty(G): M(\phi) \in (L^1(G) \otimes_{\alpha'} L^1(G))^*\}$; clearly

$$B(G) \subseteq \mathscr{W}^\gamma(G) \subseteq \mathscr{W}^\alpha(G) \subseteq \mathscr{W}^\lambda(G) = L^\infty(G).$$

When $\alpha=\alpha_{p'q'}$, $1 \leqq p \leqq q \leqq \infty$, we write $\mathscr{V}^{pq}(G)$, $\mathscr{L}^{qp}(G)$, $\mathscr{W}^{pq}(G)$.

THEOREM 6. *If $V^\alpha(G,G)$ is a Varopoulos algebra, then $\mathscr{V}^\alpha(G)$ and $\mathscr{W}^\alpha(G)$ are Banach algebras under pointwise multiplication. In particular, $\mathscr{V}^{pq}(G)$ always is such a Banach algebra. For arbitrary tensorial norm α, $\mathscr{V}^\alpha(G)$ and $\mathscr{W}^\alpha(G)$ are Banach $\mathscr{V}^\gamma(G)$-modules while*

$$\mathscr{W}^\alpha(G) \cap \mathscr{C}(G) = \mathscr{W}^\alpha(G_d) \cap \mathscr{C}(G) \qquad \text{"Bochner-Eberlein"}$$

isometrically ($G_d=G$ with discrete topology).

The most precise results are obtained when G is amenable with an interesting use of the Glicksberg-Reiter theorem.

THEOREM 7. *Let G be an amenable group. Then, up to equivalence of norms,*

$$\mathscr{W}^\alpha(G) = (\mathscr{L}^{\alpha'}(G))^*, \qquad \mathscr{W}^\alpha(G) \cap \mathscr{C}(G) = (\mathscr{L}_d^{\alpha'}(G))^* \cap \mathscr{C}(G),$$

where $\mathscr{L}_d^{\alpha'}(G)$ denotes the closure of $l^1(G_d)$ in $(\mathscr{V}^\alpha(G))^*$.

4. **Applications to L^p-convolution operator theory.** Using characterizations of $(\mathscr{C}_0(X) \otimes_\alpha \mathscr{C}_0(Y))^*$ in terms of (p,q)-*absolutely summing* operators stemming from Theorem 4, together with characterizations of $(L^1(G) \otimes_{\alpha'} L^1(G))^*$, $\alpha=\alpha_{p'q'}$, in terms of (p,q)-*integral* operators, we obtain

THEOREM 8. *For any locally compact group G the following inclusions hold:*

(i) $A^p(G) \subseteq \mathscr{V}^{pp}(G)$, $1 \leqq p \leqq \infty$;

(ii) $B(G) \subseteq \mathscr{W}^{pp}(G) \subseteq B^p(G)$, $1 < p < \infty$;

except possibly for the first inclusion in (ii) *all embeddings are norm-decreasing.*

Theorems 2, 5 and 6 now provide a completely new approach to the main results of Herz (both in [6] and unpublished) since $\mathscr{V}^{pp}(G)$ is a closed subspace of $\mathscr{W}^{pp}(G)$.

THEOREM 9. *For each locally compact group G and each p, $1 \leqq p \leqq \infty$, $A^p(G)$ is a Banach algebra and a Banach $A^q(G)$-module via pointwise multiplication when $1 \leqq p \leqq q \leqq 2$ or $2 \leqq q \leqq p \leqq \infty$.*

These algebras $\mathscr{V}^{pq}(G)$ have useful identifications. The Banach space of (right-) convolution operators $T: L^r(G) \to L^s(G)$ will be denoted by $Cv^{rs}(G)$, the closure of $L^1(G)$ in $Cv^{rs}(G)$ by $cv^{rs}(G)$ (when r, s are suitably restricted). In case $r=\infty$ or $s=\infty$, $\mathscr{C}_0(G)$ replaces $L^\infty(G)$. It is known that, when G is amenable, $Cv^{pq}(G)=(A^{pq}(G))^*$ isometrically when

$$A^{pq}(G)= P(L^{p'}(G) \otimes_\gamma L^q(G))$$

(cf. [1]).

THEOREM 10. *Let G be an amenable group. Then isometrically*
(i) $A^p(G)=\mathscr{V}^{pp}(G), \quad 1 \leqq p \leqq \infty$;
(ii) $cv^p(G)=\mathscr{L}^{pp}(G), \quad 1 \leqq p \leqq \infty$;
(iii) $B^p(G)=\mathscr{W}^{pp}(G), \quad 1<p<\infty$;
(iv) $(cv^p(G))^*=B^p(G), \quad 1<p<\infty$.
In particular,

$$A(G) \subseteq A^q(G) \subseteq A^p(G) \subseteq \mathscr{C}_0(G)$$

whenever $1 \leqq p \leqq q \leqq 2$ *or* $2 \leqq q \leqq p \leqq \infty$.

THEOREM 11. *Let G be a compact group. Then isometrically*
(i) $A^{pq}(G)=\mathscr{V}^{pq}(G)$,
(ii) $cv^{qp}(G)=\mathscr{L}^{qp}(G)$
for $1 \leqq p \leqq q \leqq \infty$.

Part (iii) of Theorem 5 when translated into convolution operator theory says that

$$(2) \qquad Cv^{pr}(G) = Cv^{p1}(G), \qquad G \text{ compact}, \quad 1 \leqq r < p < 2.$$

Doss established (2) for compact abelian groups by showing that $Cv^{qp}(G)$ coincides with the space $Cv^p_\omega(G)$ of *weak type* (p, p) convolution operators. Setting $A^p_\omega(G)=P(L^{p'1}(G) \otimes_\gamma L^p(G))$, G abelian, $1<p<\infty$, we can complete the identification of the algebras $\mathscr{V}^{pq}(G)$, G abelian.

THEOREM 12. *Let G be a locally compact abelian group. Then*

$$\mathscr{V}^{rp}(G) = A^p_\omega(G), \qquad \mathscr{L}^{pr}(G) = cv^p_\omega(G), \qquad (\mathscr{V}^{rp}(G))^* = Cv^p_\omega(G)$$

provided $1 \leqq r<p<2$. *In particular,* $A^p_\omega(G)$ *is a pointwise Banach algebra.*

The techniques which this approach provides lead to solutions for arbitrary amenable groups of many of the problems left open by Eymard [1]. In addition, nine of the squares left open in the multiplier table given by Hewitt-Ross [7, pp. 410–411] can be completed and partial information given for the remaining two.

REFERENCES

1. P. Eymard, *Algèbres A^p et convoluteurs de L^p*, Séminaire Bourbaki: 1969/70, Exposé 367.
2. J. E. Gilbert, *Tensor products of Banach spaces and norm ideals of operators*, Lecture notes, Universität Heidelberg, 1973.
3. ———, *L^q-convolution operators and tensor products of Banach spaces.* I, II, III (submitted for publication).
4. Y. Gordon, D. R. Lewis and J. R. Retherford, *Banach ideals of operators with applications*, J. Functional Analysis **14** (1973), 85–129.
5. A. Grothendieck, *Résumé de la théorie métrique des produits tensoriels topologiques*, Bol. Soc. Mat. São Paulo **8** (1953), 1–79. MR **20** #1194.
6. C. S. Herz, *The theory of p-spaces with an application to convolution operators*, Trans. Amer. Math. Soc. **154** (1971), 69–82. MR **42** #7833.
7. E. Hewitt and K. A. Ross, *Abstract harmonic analysis.* Vol. II. *Structure and analysis for compact groups analysis on locally compact Abelian groups*, Die Grundlehren der math. Wissenschaften, Band 152, Springer-Verlag, Berlin and New York, 1970. MR **41** #7378.
8. J.-T. Lapresté, *Sur une généralisation de la notion d'opérateurs nucléaires et sommantes dans les espaces de Banach*, C.R. Acad. Sci. Paris Sér. A-B **275** (1972), A45-A48. MR **46** #2468.
9. J. Lindenstrauss and A. Pełczyński, *Absolutely summing operators in $\mathscr{L}_p$-spaces and their applications*, Studia Math. **29** (1968), 275–326. MR **37** #6743.
10. H. P. Rosenthal, *On subspaces of L^p*, Ann. of Math. (2) **97** (1973), 344–373. MR **47** #784.
11. N. Th. Varopoulos, *Tensor algebras and harmonic analysis*, Acta Math. **119** (1967), 51–112. MR **39** #1911.
12. ———, *Tensor algebras over discrete spaces*, J. Functional Analysis **3** (1969), 321–335. MR **40** #3328.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TEXAS, AUSTIN, TEXAS 78712