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PRODUCTS OF KNOTS

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0. **Introduction.** Let $f:C^n \rightarrow C$ be a (complex) polynomial mapping with an isolated singularity at the origin of C^n. That is, $f(0)=0$ and the complex gradient df has an isolated zero at the origin. The *link* of this singularity is defined by the formula $L(f)=V(f)\cap S^{2n-1}$. Here the symbol $V(f)$ denotes the variety of f, and S^{2n-1} is a sufficiently small sphere about the origin of C^n.

Given another polynomial $g:C^m \rightarrow C$, form $f+g$ with domain $C^{n+m}=C^n\times C^m$ and consider $L(f+g)\subset S^{2n+2m-1}$.

In this note, we announce a topological construction for $L(f+g)\subset S^{2n+2m-1}$ in terms of $L(f)\subset S^{2n-1}$ and $L(g)\subset S^{2m-1}$. The construction generalizes the algebraic situation. Given nice codimension-two imbeddings $K\subset S^n$ and $L\subset S^m$, we form a product $K\otimes L\subset S^{n+m+1}$. Then $L(f)\otimes L(-g)\simeq L(f+g)$.

§1 outlines the construction and its properties. §2 gives applications to iterated branched covering constructions, knot theory, and orthogonal group actions.

This construction and the results of §1 have also been found independently by W. Neumann [7].

1. **The construction of products.** All manifolds will be smooth. Each ambient sphere S^n comes equipped with an orientation.

A *knot* in S^n is any closed oriented codimension-two submanifold K. Given a knot $K\subset S^n$ we may write $S^n=E_K\cup(K\times D^2)$ where E_K is a

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manifold with boundary equal to $K\times S^1$. If n is larger than 3, we assume that K is connected. Thus, by Alexander duality, $H^1(E_K)\simeq Z$. One may choose $\phi:E_K\to S^1$ representing the generator of $H^1(E_K)$ so that ϕ is differentiable and $\phi|\partial E_K$ is projection on the second factor. If $n=3$, then K may consist of a collection of disjointly imbedded circles. A choice of orientations for these circles determines ϕ so that ϕ^{-1} (regular value) is an oriented spanning surface for K which induces the chosen orientations on each component.

A knot is said to be *spherical* if it is homeomorphic to a sphere.

A knot is said to be *fibered* if there is a choice of ϕ as above so that $\phi:E_K\to S^1$ is a locally trivial smooth fibration.

Now suppose that we are given knots $K\subset S^n$ and $L\subset S^m$ and corresponding maps $\phi:E_K\to S^1$ and $\psi:E_L\to S^1$. If one knot is fibered, then $E_K\times_{S^1}E_L=\{(x,y)\in E_K\times E_L|\phi(x)=\psi(y)\}$ is a well-defined smooth manifold with boundary. Henceforth, when dealing with a pair of knots, we shall assume that at least one knot is fibered. We now define a manifold $K\otimes L$ and, using its properties, obtain the product knot $K\otimes L\subset S^{n+m+1}$.

DEFINITION. Given knots K and L as above, define the manifold

$$K\otimes L\simeq (K\times D^{m+1})\cup(E_K\times_{S^1}E_L)\cup(D^{n+1}\times L).$$

These three pieces are attached according to the following description. Note that

$$\partial(E_K\times_{S^1}E_L)\simeq(K\times E_L)\cup(E_K\times L)$$

and

$$\partial(K\times D^{m+1})\simeq(K\times D^2\times L)\cup(K\times E_L),$$
$$\partial(D^{n+1}\times L)\simeq(K\times D^2\times L)\cup(E_K\times L).$$

Using these boundary identifications, glue the three pieces together to form a closed manifold.

LEMMA 1. *Given K and L as above the manifold $K\otimes L$ is independent of the choices of maps ϕ and ψ used in its construction. If either K or L is isotopic to a trivial knot (say $L=S^{m-2}\subset S^m$, the standard imbedding), then $K\otimes L\simeq S^{n+m-1}$.*

Now given $\phi:E_L\to S^1$, there is an imbedding $\hat{\phi}:E_L\to D^{m+1}\times S^1$ (essentially $\hat{\phi}(x)=(x,\phi(x))$). This induces an embedding $K\otimes L\subset K\otimes S^m\simeq S^{m+n+1}$ (by Lemma 1).

LEMMA 2. *The imbedding $K\otimes L\subset S^{n+m+1}$ described in the previous paragraph is well defined up to ambient isotopy. If one chooses the imbedding induced by the inclusion $S^n\subset S^{n+2}$, then the resulting knot $L\otimes K\subset S^{n+m+1}$ is ambient isotopic to $K\otimes L$. Thus the product of two knots is now defined.*

Given polynomials f and g as in the introduction, we have $L(f)\subset S^{2n-1}$ and $L(g)\subset S^{2m-1}$. These are fibered knots. The maps to S^1 are given by the Milnor fibering (see [5]).

THEOREM 3. *There is a diffeomorphism $L(f+g)\simeq L(f)\otimes L(-g)$. Furthermore, $L(f+g)\subset S^{2n+2m-1}$ (naturally) and $L(f)\otimes L(-g)\subset S^{2n+2m-1}$ (by Lemma 2). These imbeddings are ambient isotopic.*

Note that, up to orientations, $L(f)\otimes L(-g)$ is diffeomorphic to $L(f)\otimes L(g)$. However, the imbeddings may differ by an orientation reversing diffeomorphism of the containing sphere.

Suppose that $K\subset S^n$ is any knot and that $L\subset S^m$ is a fibered knot. Let $F\subset S^n$ be an orientable spanning surface for K.

THEOREM 4. *Under the above conditions there is an oriented manifold $F\otimes L\subset S^{n+m+1}$ such that $\partial(F\otimes L)\simeq K\otimes L$. This imbedding of $F\otimes L$ induces the given imbedding of $K\otimes L$. The manifold $F\otimes L$ has the homotopy type of $F * F'$ ($*$ denotes join) where F' is a spanning surface for L in S^m. In fact, $F * F'\to F\otimes L$ so that the induced map $F * F'\to S^{n+m+1}$ is, up to homotopy, $F * F'\subset S^n * S^m$.*

COROLLARY 5. *If $K\subset S^n$ and $L\subset S^m$ have spanning surfaces F and F' as above and corresponding Seifert matrices (see [4]) V and V', then $F\otimes L\subset S^{n+m+1}$ has Seifert matrix $\pm V\otimes V'$. (Here the symbol $\otimes$ denotes the tensor product of matrices.)*

2. **Applications.** (a). The construction of $K\otimes L$ applies to the empty knots in S^1. That is, let $[a]$ denote the empty knot with $E_{[a]}=S^1$ and $\phi: S^1\to S^1$ given by the formula $\phi(x)=x^a$. Our methods still apply, even though ϕ is a multiple of the generator of $H^1(S^1)$. Thus

$$K\otimes [a]\simeq (K\times D^2)\cup (E_K\times_{S^1} E_{[a]}).$$

This is the a-fold cyclic branched cover of S^n with branch set K. By the method of Lemma 2 there is an imbedding $K\otimes[a]\subset S^{n+2}$. This construction may be iterated to form $K\otimes[a_1]\otimes[a_2]\otimes\cdots\otimes[a_s]\subset S^{n+2s}$ for any knot $K\subset S^n$ and a given sequence of positive integers $\{a_1, a_2, \cdots, a_s\}$. This iterated branched covering was discussed in [3] for fibered knots K. It has been noted for arbitrary K by W. Neumann in [6].

In particular, $[a_1]\otimes[a_2]\otimes\cdots\otimes[a_s]$ is diffeomorphic to the Brieskorn manifold $\sum(a_1, a_2, \cdots, a_s)=L(z_1^{a_1}+\cdots+z_s^{a_s})$. It follows that $K\otimes[a_1]\otimes\cdots\otimes[a_s]$ is diffeomorphic to $K\otimes\sum(a_1,\cdots, a_s)$.

(b) Let $\Lambda\subset S^3$ denote the link of two trivial circles with linking number equal to $+1$. Then Λ bounds an annulus and has Seifert matrix $V'=(1)$. Since Λ is fibered, we may form $K\otimes\Lambda\subset S^{n+4}$ for any knot $K\subset S^n$. If K

has Seifert matrix V then $K\otimes\Lambda$ has Seifert matrix $\pm V$. Hence we conclude as follows.

THEOREM 6. *Let C_s denote the Levine knot cobordism group (see* [4]*) of spherical knots in S^{2n+1} ($n\geqq 3$, $s=2n-1$). Define $\omega: C_s\to C_{s+4}$ by $\omega(K)=K\otimes\Lambda$. Then ω is an isomorphism.*

REMARKS. Note that Λ is the Brieskorn manifold $\Sigma(2, 2)$. Thus $\omega(K)$ is obtained by two double branched coverings as in (a). In fact, our description of ω coincides with that given by G. Bredon in [1]. To see this, we construct $O(m)$ actions on knot products.

(c) Let $\Lambda_m=L(z_1^2+z_2^2+\cdots+z_m^2)$. Then the pair (S^{2m-1}, Λ_m) has a natural $O(m)$ action so that Λ_m is an orbit. It is then easy to see the following.

PROPOSITION 7. *Let $K\subset S^n$ be an arbitrary knot. Then $K\otimes\Lambda_m$ admits a smooth $O(m)$ action with three orbit types and orbit space D^{n+1}. The set of fixed points corresponds to $K\subset\partial D^{n+1}$.*

This gives a new construction for these $O(m)$ manifolds. Combined with Theorem 6, it shows that our isomorphism is the same as that of Bredon. Various properties of $O(m)$ manifolds follow easily from this viewpoint.

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