BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 80, Number 5, September 1974

ON THE EXTENSION OF BOUNDARY INTEGRABLE ALMOST COMPLEX STRUCTURE[1]

BY GARO K. KIREMIDJIAN

Communicated by I. M. Singer, March 6, 1974

1. **Introduction.** Let $\{M, M'\}$ be a finite Kähler manifold, i.e., M' is a complex Kähler manifold, M is an open submanifold of M' with compact closure, $M_0=bM$, the boundary of M, is a C^∞ submanifold of M', and for each $p \in M_0$ there exists a coordinate neighborhood U of p with real coordinates $t^1, \cdots, t^{2n-1}$, r such that $r(q)<0$ for $q \in U \cap M$ and $r(q)>0$ for $q \in U \cap (M'-M)$. It is assumed that the following conditions hold:

A. For each boundary point the Levi form has at least two positive eigenvalues.

B. There exists a constant $c_0>0$ such that for all $u \in C^{0,q}(\bar{M}, \Theta)$, $q=1, 2$ $((2\square-\triangle)u, u) \geqq c_0(u, u)$ where Θ is the holomorphic tangent bundle of M', $C^{p,q}(\bar{M}, \Theta)$ is the space of all C^∞ Θ-valued (p, q)-forms extendible to a neighborhood of $\bar{M}$, $\square$ (resp., $\triangle$) is the complex (resp., the real) Laplacian on $C^{p,q}(\bar{M}, \Theta)$ and $(\ ,\ )$ is the L_2-inner product over M (see [2]).

Then the main result of this note states that a sufficiently small integrable almost-complex structure on M_0 can be extended to a complex structure on M. A complete proof will appear elsewhere; a brief outline follows.

However, we first take a closer look at condition B. Let D be the covariant differentiation operator associated with the connection θ of the metric g on M', i.e.,

$$Du = du + \theta \wedge u = \bar{\partial} u + \tilde{\partial} u$$

for $u \in C^{p,q}(\bar{M}, \Theta)$. Let D^* and $\bar{\partial}^*$ be the formal adjoints of D and $\bar{\partial}$, respectively. Then $\triangle = DD^* + D^*D$ and $\square = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$. Since g is Kähler, $\triangle = 2\square - K$, $K = \sqrt{-1}(e(s)\Lambda - \Lambda e(s))$, where

$$e(s)u = \bar{\partial}\theta \wedge u, \qquad \Lambda u = *^{-1}(\rho \wedge * u),$$

$*$ is the Hodge star operator and ρ is the Kähler form of g. We refer

---

AMS (MOS) subject classifications (1970). Primary 32G99; Secondary 32G05, 35J60, 32J25.

[1] This research was supported in part by NSF grant GP 33942X1.

to [**3**, pp. 482–483], for verification of this identity. Hence, condition B requires the existence of a constant $c_0>0$ such that $(Ku, u)\geqq c_0(u, u)$ for all $u \in C^{0,q}(\bar{M}, \Theta)$, $q=1, 2$. Now it is established in [**2**, p. 276], that if the scalar curvature is sufficiently negative, then one has the stronger result $\langle Ku, u\rangle_x \geqq c_0\langle u, u\rangle_x$ for all $x \in M'$, where $\langle\ ,\ \rangle_x$ is the inner product at the point x, i.e., Θ is $W^{0,q}$-elliptic. It is also shown in [**2**] that the criterion of W-ellipticity is satisfied for a large class of bounded homogeneous domains in C^n provided with the Bergman metric. More generally, let M' be a manifold whose universal covering space $\tilde{M}'$ is isomorphic to $D_1\times\cdots\times D_r$, where D_i is a bounded irreducible symmetric domain with $\dim_C D_i\geqq 3$. Then Θ is $W^{0,q}$-elliptic for $0\leqq q\leqq 2$, and condition B will hold for any relatively compact open submanifold M of M' with smooth boundary.

2. **Definitions and notation.** Let M_0 be a C^∞ manifold of real dimension $2n-1$ and let CTM_0 be the complexified tangent bundle.

2.1. DEFINITION. An almost-complex structure on M_0 is given by a complex subbundle E'' of CTM_0 of fiber complex dimension $n-1$ such that $E''\cap\bar{E}''=\{0\}$.

2.2. DEFINITION. The almost-complex structure E'' on M_0 is integrable if, for any two sections L and L' of E'' over an open set U of M_0, $[L, L']$ is also a section of E''.

We now assume that M_0 is the boundary of a finite complex manifold $\{M, M'\}$. The complex structure on M' induces an integrable almost-complex structure T'' on M_0.

2.3. DEFINITION. The almost-complex structure E'' on M_0 is of finite distance from T'' if $\pi''|E'':E''\to T''$ is an isomorphism where $\pi'':CTM_0\to T''$ is the projection.

In this case $E''=\{X-\tau\circ\varphi(X)|X\in T''\}$ where $\tau:\Theta|M_0\to T'\oplus CF$ is an isomorphism, $T'=\bar{T}''$, CF is the complexification of a real one-dimensional subbundle F of TM_0 such that $CTM_0=T'\oplus T''\oplus CF$ and

$$\varphi = -\tau^{-1}\circ(\mathrm{id}-\pi'')\circ(\pi''E'')^{-1}:T''\to\Theta\ |\ M_0,$$

i.e., φ is a $\Theta|M_0$-valued C^∞ differential form on M_0 of type $(0, 1)_b$. Conversely, any such differential form φ gives rise to an almost complex structure E'' on M_0. We will denote E'' by T''_φ. As in the case of complex manifolds, there exists a $\Theta|M_0$-valued C^∞ differential form Φ on M_0 of type $(0, 2)_b$ such that $\Phi=0$ if and only if T''_φ is integrable.

Let φ be a T'-valued form and let $\omega\in C^{0,1}(\bar{M}, \Theta)$ be such that $t\omega$, the complex tangential part of ω, is equal to φ on M_0. Let $\Omega=\bar{\partial}\omega-[\omega, \omega]$. If T''_ω is the almost complex structure on M induced by ω, then one can show that $T''_\varphi=CTM_0\cap T''_\omega$ and $t\Omega=0$ on M_0 if and only if $\Phi=0$.

3. **The main result.** Now we can formulate the following extension problem.

THEOREM. *Let $\{M, M'\}$ be a finite complex Kähler manifold such that conditions* A *and* B *in* §1 *are satisfied. Let φ be a T'-valued C^∞ differential form of type* $(0, 1)_b$ *with sufficiently small Hölder norm* $|\varphi|_{k+\alpha}$, $0<\alpha<1$, *for some integer $k>0$ depending on n. Assume that T''_φ is integrable. Then there exists $\omega \in C^{0,1}(\bar{M}, \Theta)$ such that $\Omega=0$ and $t\omega=\varphi$ on M_0.*

We first consider the quadratic form

$$Q(u, v) = \tfrac{1}{2}[(Du, Dv) + (D^*u, D^*v) + (Ku, v)] - 2([\psi, u], \bar{\partial}v)$$

for some $\psi \in C^{0,1}(\bar{M}, \Theta)$ with sufficiently small norm and $u, v \in \mathfrak{B} = \{\omega \in C^{0,1}(\bar{M}, \Theta) | t\omega=0 \text{ on } M_0\}$. One can easily check that by condition B, const $N^2(u) \leq |\text{Re } Q(u, u)| \leq$ const $N^2(u)$ where Re stands for the real part of $Q(u, u)$, and $N^2(u) = \|u\|^2 + \|Du\|^2 + \|D^*u\|^2$. Hence, if $\|u\|_s$ is the Sobolev s-norm of u, then $\|u\|_1 \leq \text{const}|\text{Re } Q(u, u)|$.

It follows from the theory developed in [1] and [4] that for each $\sigma \in C^{0,1}(\bar{M}, \Theta)$ there exists a unique $u \in \mathfrak{B}$ such that $Q(u, v) = (\sigma, v)$ for all $v \in \hat{\mathfrak{B}}$, the completion of $\mathfrak{B}$ with respect to the norm N such that

$$\|u\|_{s+2} \leq c_s \|\sigma\|_s; \tag{1}$$

$$L_\psi u = \tfrac{1}{2}(DD^* + D^*D + K)u - 2\bar{\partial}^*[\psi, u] = \sigma; \tag{2}$$

$$tD^*u = 0 \quad \text{on } M_0; \tag{3}$$

$$|u|_{k+\alpha+2} \leq c'_k |f|_{k+\alpha} \tag{4}$$

for sufficiently large k. The constants c_s and c'_k depend on s and k and on the derivatives of ψ up to order s and k, respectively. If $|\psi|_{k+\alpha}$ is sufficiently small we may assume that c'_k in (4) depends only on k.

We observe that $D^*u = -* \bar{\partial} * u - * \tilde{\partial} * u$, and since u is a form of type $(0, 1)$, $\tilde{\partial} * u = 0$ and $D^*u = \bar{\partial}^*u$. On the other hand for a Kähler metric g the complex Laplacian $\Box = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ is $\tfrac{1}{2}(DD^* + D^*D + K)$, and if $\sigma = \bar{\partial}^*h$ for $h \in C^{0,2}(\bar{M}, \Theta)$, then (3) and Stokes' theorem imply that $L_\psi u = \sigma$ if and only if

$$\bar{\partial}^* \bar{\partial}u - 2\bar{\partial}^*[\psi, u] = \bar{\partial}^*h. \tag{5}$$

We now consider the nonlinear differential system $\bar{\partial}^*\Omega = 0$. Let $\omega_0 \in C^{0,1}(\bar{M}, \Theta)$ be an extension of φ such that $|\omega_0|_{k+\alpha} \leq \text{const}|\varphi|_{k+\alpha}$. One can inductively construct a sequence of approximate solutions $\omega_{m+1} = \omega_m + u_m$, where u_m is the solution of (5) with $tu_m = t\bar{\partial}^*u_m = 0$ on M_0, $\psi = \omega_m$, $h = -\Omega_m = -\bar{\partial}\omega_m + [\omega_m, \omega_m]$. Since $|\bar{\partial}^*\Omega_m|_{k+\alpha-2} \leq \text{const}|u_m|_{k+\alpha}$, (4) implies that there exists a constant $c>0$ such that $|\omega_{m+1} - \omega_m|_{k+\alpha} \leq c|\omega_m - \omega_{m-1}|^2_{k+\alpha}$

for $m=1, 2, \cdots$. This is enough to conclude that there exists a Θ-valued form ω of type (0, 1) and of class $C^{k+\alpha}$ on $\bar{M}$ such that $\bar{\partial}^*\Omega=0$, $t\omega=\varphi$ on M_0, and $|\omega|_{k+\alpha}\leqq\text{const}|\varphi|_{k+\alpha}$.

Now it can easily be shown that $\bar{\partial}\Omega=2[\omega, \Omega]$. By condition A and the fact that the normal part of $*\,\#\Omega$ vanishes on M_0, the basic estimate of the $\bar{\partial}$-Neumann problem holds for $*\,\#\Omega$, i.e.,

$$E(*\,\#\Omega) \leqq \text{const}(\|\Omega\|^2 + \|\bar{\partial}\Omega\|^2 + \|\bar{\partial}^*\Omega\|^2).$$

For the definition of the norm E, we refer to [5] and [6]; the operators $*$ and $\#$ are defined in [2]. Then by condition B and the complete continuity of E, one can obtain the estimate $\|\bar{\partial}\Omega\|\leqq c_0|\omega|_{1,\alpha}\|\bar{\partial}\Omega\|$ for some constant c_0. Thus $\bar{\partial}\Omega=0$ if $|\varphi|_{k+\alpha}$ is sufficiently small. Since $t\Omega=0$ on M_0, $\bar{\partial}\Omega=0$, and $\bar{\partial}^*\Omega=0$, condition B implies that $\Omega=0$.

Finally, it follows from the construction of approximate solutions that $\omega=\omega_0+w$, where w is of class $C^{k+\alpha}$ and $\bar{\partial}^*w=0$. Then $\bar{\partial}^*(\bar{\partial}\omega-[\omega, \omega])=0$ can be expressed as

$$\Box w - \bar{\partial}^*(2[\omega_0, w] + [w, w]) = \bar{\partial}^*([\omega_0, \omega_0] - \bar{\partial}\omega_0).$$

This equation is elliptic if $|\varphi|_{k+\alpha}$ is sufficiently small. Since ω_0 is of class C^∞, w is also of class C^∞.

References

1. S. Agmon, A. Douglis and L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions*. II, Comm. Pure Appl. Math. **17** (1964), 35–92. MR **28** #5252.

2. A. Andreotti and E. Vesentini, *On deformations of discontinuous groups*, Acta Math. **112** (1964), 249–298. MR **29** #6503.

3. E. Calabi and E. Vesentini, *On compact, locally symmetric Kähler manifolds*, Ann. of Math. (2) **71** (1960), 472–507. MR **22** #1922b.

4. G. Fichera, *Linear elliptic differential systems and eigenvalue problems*, Lecture Notes in Math., no. 8, Springer-Verlag, Berlin and New York, 1965. MR **35** #536.

5. J. J. Kohn, *Harmonic integrals on strongly pseudo-convex manifolds*. I, Ann. of Math. (2) **78** (1963), 112–148. MR **27** #2999.

6. J. J. Kohn and H. Rossi, *On the extension of holomorphic functions from the boundary of a complex manifold*, Ann. of Math. (2) **81** (1965), 451–472. MR **31** #1399.

Department of Mathematics, Stanford University, Stanford, California 94305