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ZEROS OF DERIVATIVE OF RIEMANN'S ξ-FUNCTION

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Riemann's ξ-function is defined by $\xi(s)=H(s)\zeta(s)$ where $\zeta(s)$ is the Riemann zeta-function and $H(s)=\frac{1}{2}(s^2-s)\pi^{-s/2}\Gamma(s/2)$. The functional equation is $\xi(s)=\xi(1-s)$. Moreover ξ is an entire function which has as its zeros precisely those of ζ in the critical strip. Because $\xi(\frac{1}{2}+it)$ is real, it follows that between consecutive zeros of ξ on the $\frac{1}{2}$-line there is at least one zero of ξ'.

It has recently been shown [1], [2] that $\zeta(s)$ has at least $\frac{1}{3}$ of its zeros in the critical strip on $\sigma=\frac{1}{2}$. Here a similar result will be proved for $\xi'(s)$ (for which $\frac{1}{3}$ is already implied by the remark above). Let $U=T/\log^{10} T$. Then the following theorem will be sketched.

THEOREM. *More than $\frac{7}{10}$ of the zeros of $\xi'(s)$ in $T<t<T+U$ occur on $\sigma=\frac{1}{2}$.*

By Stirling's formula, for $|\sigma|<10$, $H(s)=e^{F(s)}$, where

$$F'(s) = \tfrac{1}{2}\log s/2\pi + O(1/s), \qquad F''(s) = O(1/s).$$

From $\xi(s)=H(s)\zeta(s)=H(1-s)\zeta(1-s)$ follows

$$\begin{aligned} \xi'(s) &= H'(s)\zeta(s) + H(s)\zeta'(s) \\ &= -H'(1-s)\zeta(1-s) - H(1-s)\zeta'(1-s), \end{aligned} \tag{1}$$

and also

$$\begin{aligned} &H''(s)\zeta(s) + 2H'(s)\zeta'(s) + H(s)\zeta''(s) \\ &\quad = H''(1-s)\zeta(1-s) + 2H'(1-s)\zeta'(1-s) + H(1-s)\zeta''(1-s). \end{aligned}$$

Since $H'=HF'$, and $H''=H'F'+HF''$,

$$\begin{aligned} &F'(s)[H'(s)\zeta(s) + H(s)\zeta'(s)] \\ &\qquad\qquad - F'(1-s)[H'(1-s)\zeta(1-s) + H(1-s)\zeta'(1-s)] \\ &\quad = -H'(s)\zeta'(s) - H(s)\zeta''(s) - H(s)F''(s)\zeta(s) \\ &\qquad + H'(1-s)\zeta'(1-s) + H(1-s)\zeta''(1-s) \\ &\qquad + H(1-s)F''(1-s)\zeta(1-s). \end{aligned}$$

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By (1) this can be written as

$$\begin{aligned}(2)\qquad &(F'(s)+F'(1-s))\xi'(s)\\ &\quad = -H(s)[F'(s)\zeta'(s)+\zeta''(s)+F''(s)\zeta(s)]\\ &\qquad + H(1-s)[F'(1-s)\zeta'(1-s)+\zeta''(1-s)+F''(1-s)\zeta(1-s)].\end{aligned}$$

From the functional equation, $-2\xi'(s)=-\xi'(s)+\xi'(1-s)$, and so

$$\begin{aligned}&-2\xi'(s)(F'(s)+F'(1-s))\\ &\quad = -(F'(s)+F'(1-s))(H'(s)\zeta(s)+H(s)\zeta'(s))\\ &\qquad + (F'(s)+F'(1-s))(H'(1-s)\zeta(1-s)+H(1-s)\zeta'(1-s)).\end{aligned}$$

Adding the above to (2) gives

$$\begin{aligned}(3)\qquad &-(F'(s)+F'(1-s))\xi'(s)\\ &\quad = -H(s)[F'(s)\zeta'(s)+\zeta''(s)+F''(s)\zeta(s)]\\ &\qquad - H(s)[F'(s)+F'(1-s)][F'(s)\zeta(s)+\zeta'(s)]\\ &\qquad + H((1-s)[F'(1-s)\zeta'(1-s)+\zeta''(1-s)+F''(1-s)\zeta(1-s)])\\ &\qquad + H(1-s)[F'(s)+F'(1-s)][F'(1-s)\zeta(1-s)+\zeta'(1-s)].\end{aligned}$$

Let

$$\begin{aligned}G(s) &= \zeta(s)+\zeta'(s)/F'(s)\\ &\quad + [F'(s)+F'(1-s)]^{-1}(\zeta'(s)+\zeta''(s)/F'(s)+F''(s)\zeta(s)/F'(s)).\end{aligned}$$

Then (3) becomes

$$\xi'(s) = F'(s)H(s)G(s) - F'(1-s)H(1-s)G(1-s).$$

For $s=\frac{1}{2}+it$, the right side above is the difference between two complex conjugate quantities. Hence $\xi'(\frac{1}{2}+it)=0$, where

$$\arg(F'HG(\tfrac{1}{2}+it)) \equiv 0 \pmod{\pi}.$$

Since $F'\sim(\log t/2\pi)/2$, it has little effect on the change in argument as t increases. By Stirling's formula $\arg H(\frac{1}{2}+it)$ changes rapidly and by itself would supply the full quota of zeros of $\xi'(s)$ on $\sigma=\frac{1}{2}$. However G also plays a role. What will be shown is that the change in $\arg G$ is sufficiently restricted so that it cancels less than 30% of the change in $\arg H$.

To get the change in $\arg G(\frac{1}{2}+it)$, the principle of the argument can be used. The determination of the number of zeros of $G(s)$ in a rectangle D with vertices $(\frac{1}{2}+iT, 3+iT, \frac{1}{2}+i(T+U), 3+i(t+U))$ leads in a familiar way to the change in $\arg G$ on $\sigma=\frac{1}{2}$, $T<t<T+U$. To get the number of zeros of G in D, Littlewood's lemma [**3**, §9.9] is used in a familiar way [**3**, §9.15]. However it turns out to be more efficient to first multiply G by an

entire function $\psi(s)$ even though this may introduce extra zeros. The key term in the estimate of the number of zeros of ψG in D is

$$\int_T^{T+U} \log|\psi G(a+it)|\,dt/2\pi(\tfrac{1}{2}-a),$$

where $a<\frac{1}{2}$ and $\frac{1}{2}-a$ is small. Use is now made of

$$\int_T^{T+U} \log|\psi G(a+it)|\,dt \leqq \frac{U}{2}\log\left(\frac{1}{U}\int_T^{T+U}|\psi G(a+it)|^2\,dt\right).$$

The choice for ψ is

$$\psi(s)=\sum_{j\leqq y}\frac{\mu(j)\log y/j}{j^{1/2-a}\log y}\frac{1}{j^s},$$

and $y=T^{1/2}/\log^{20} T$. To compute

$$J=\frac{1}{U}\int_T^{T+U}|\psi G(a+it)|^2\,dt$$

it is necessary to express ζ and its derivatives in terms of the approximate functional equation as done in [1], [2]. Let $R=(\frac{1}{2}-a)\log T/2\pi$. Then lengthy calculations lead to

$$\begin{aligned} J = {} & e^{2R}\left(\frac{1}{24R}-\frac{1}{12R^2}+\frac{2}{3R^3}-\frac{3}{R^4}+\frac{6}{R^5}\right) \\ & -\frac{R}{12}+\frac{3}{4}-\frac{29}{24R}-\frac{13}{4R^2}-\frac{20}{3R^3}-\frac{9}{R^4}-\frac{6}{R^5}. \\ & +O\left(\frac{(\log\log T)^{10}}{\log T}\right). \end{aligned}$$

For $R=1.1$, $J\leqq 1.3634$. Therefore

$$\begin{aligned} & \frac{1}{\frac{1}{2}-a}\int_T^{T+U}\log|\psi G(a+it)|\,dt \\ & \leqq \frac{U}{2(\frac{1}{2}-a)}\log 1.3634 = U\log T/2\pi\,\frac{\log 1.3634}{2R} \leqq 0.1414U\log T/2\pi. \end{aligned}$$

with $R=1.1$. Thus the change in $\arg G$ is at most $0.1414\,U\log T/2\pi$. By Stirling's formula the change in $\arg H(\frac{1}{2}+it)$ is essentially $\frac{1}{2}U\log T/2\pi$, and so the change in $\arg(HF'G)$ is at least $0.3586\,U\log T/2\pi$. Since the zeros occur mod π, the number is at least $0.7172\,U(\log T/2\pi)/2\pi$ which is more than 0.7 of the total number in $T<t<T+U$.

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