BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 80, Number 5, September 1974

TORSION ALGEBRAIC CYCLES, K_2, AND BRAUER GROUPS OF FUNCTION FIELDS

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Communicated by Hyman Bass, January 1, 1974

0. **Introduction.** Let F be a field, and let $H^*(F, \mu_n)$ denote the Galois cohomology of $\mathrm{Gal}(F_s/F)$ with coefficients in the group μ_n of nth roots of 1 for some fixed n prime to char F. Bass and Tate have shown that the natural pairing

$$F^*/F^{*n} \times F^*/F^{*n} = H^1(F, \mu_n) \times H^1(F, \mu_n) \xrightarrow{\text{cup product}} H^2(F, \mu_n^{\otimes 2})$$

is a symbol on F. In other words there is an induced homomorphism (nth power norm residue symbol) of the Milnor K_2 group [7], $R_{n,F}: K_2(F)/nK_2(F) \to H^2(F, \mu_n^{\otimes 2})$.

Tate showed that $R_{n,F}$ is an isomorphism where F is a global field, and asked whether an analogous result held for arbitrary fields. The situation is particularly interesting when $\mu_n \subset F$, because in this case

$$H^2(F, \mu_n^{\otimes 2}) \cong H^2(F, \mu_n) \otimes \mu_n \cong_n \mathrm{Br}(F) \otimes \mu_n$$

($\mathrm{Br}(F)$=Brauer group of F). Surjectivity of R_n implies, for example, that every division algebra with exponent n and center F is split by an *abelian* extension field of F. The question of surjectivity for R_2, for example, amounts to the classical question of whether a division algebra of exponent 2 is equivalent to a tensor product of quaternion algebras.

In this note I will consider the case F=function field of an algebraic variety X over a ground field k. I will give some partial results of an algebraic nature, and sketch some relations between Tate's question for F, and the global geometry of X. Detailed proofs are available in preprint form.

1. Algebraic results.

THEOREM (1.1). *Let $F=k(t_1, \cdots, t_r)$ be a rational function field in r variables over a field k, and let n be an integer prime to* char k. *Then the*

AMS (MOS) subject classifications (1970). Primary 14C99, 13A20, 13D15.
Key words and phrases. Algebraic cycles, K_2, Brauer group, function field, quadratic form.

maps

$$R_{n,k}: K_2(k)/nK_2(k) \to H^2(k, \mu_n^{\otimes 2}),$$
$$R_{n,F}: K_2(F)/nK_2(F) \to H^2(F, \mu_n^{\otimes 2})$$

have isomorphic kernels and isomorphic cokernels.

For example, if k is algebraically closed, $K_2(k)$ is divisible so

$$R_{n,k(t_1,\cdots,t_r)}: \frac{K^2(k(t_1, \cdots, t_r))}{nK_2(k(t_1, \cdots, t_r))} \cong_n \mathrm{Br}(k(t_1, \cdots, t_r)) \otimes \mu_n.$$

Both sides are zero for $r=1$, but not for $r \geqq 2$.

The main idea in the proof of (1.1) is the existence of exact sequences for any field F of characteristic prime to n ($X = \mathrm{Spec}\, F[t] = A_F^1$).

$$0 \to H^r(F, \mu_n^{\otimes q}) \to H^r(F(t), \mu_n^{\otimes q}) \to \coprod_{x \in X;\ \text{closed}} H^{r-1}(F(x), \mu_n^{\otimes q-1}) \to 0.$$

When $q=r=2$, this sequence can be compared with an exact sequence of Milnor

$$0 \to K^2(F) \to K^2(F(t)) \to \coprod F(x)^* \to 0,$$

and the theorem follows easily.

Recall a field F is a C_r field if every homogeneous form of degree q in $>q^r$ variables over F has a nontrivial zero.

THEOREM (1.2). *Let F be a C_2 field containing μ_{2^r} for some $r \geqq 1$. Then*

$$R_{2^r,F}: K_2(F)/2^r K_2(F) \to H^2(F, \mu_{2^r}^{\otimes 2}) \cong_{2^r} \mathrm{Br}(F) \otimes \mu_{2^r}$$

is injective.

The proof uses the theory of quadratic forms. Let $R(F)$ be the Grothendieck ring of quadratic forms on F, and let $I \subset R(F)$ be the ideal of forms of degree zero. Milnor [7] has described an isomorphism $\rho: I^2/I^3 \to K_2(F)/2K_2(F)$ which composes with the norm residue symbol $R_{2,F}$ to give the Hasse invariant,

$$\text{Hasse} = R_{2,F} \circ \rho.$$

Since F is C_2, quadratic forms are classified by their degree, determinant, and Hasse invariant, so $R_{2,F}$ is injective. The proof is completed by an induction argument on r.

2. **Global questions.** Let X be a regular algebraic k-scheme, where char k is prime to a given integer n. Let $\mathscr{K}_q$ denote the Zariski sheaf associated to the presheaf

$$U \xrightarrow[\text{open}]{} X \longmapsto K_q(\Gamma(U, \mathcal{O}_X))$$

where K_q is the qth algebraic K-functor of Grothendieck, Bass, Milnor, and Quillen.

Question (2.1). Are the Zariski cohomology groups $H^p(X, \mathscr{K}_q/n\mathscr{K}_q)$ finite for given p, q, n?

An affirmative answer for all p, q would imply, for example, that the groups ${}_nCH^q(X)$ and $CH^q(X)/nCH^q(X)$ are finite for all q, where $CH^q(X)$ denotes the Chow group of algebraic cycles mod rational equivalence on X. The argument here uses the identification

$$CH^q(X) \cong H^q(X, \mathscr{K}_q)$$

[3], [9]. For example:

PROPOSITION (2.2). *There is an exact sequence*

$$H^1(X, \mathscr{K}_2) \xrightarrow{n} H^1(X, \mathscr{K}_2) \longrightarrow H^1(X, \mathscr{K}_2/n\mathscr{K}_2) \longrightarrow {}_nCH^2(X) \longrightarrow 0.$$

Let $\mathrm{Br}'(X)$ denote the cohomological Brauer group of X [1].

THEOREM (2.3). *Let $F=k(X)$, and assume $\mu_n \subset F$. Then there is an exact sequence*

$$0 \to \mathrm{Ker}(R_{n,F}) \to \Gamma(X, \mathscr{K}_2/n\mathscr{K}_2) \to {}_n\mathrm{Br}'(X) \otimes \mu_n$$
$$\to \mathrm{Coker}(R_{n,F}) \to H^1(X, \mathscr{K}_2/n\mathscr{K}_2) \to N^1H^3_{\text{ét}}(X, \mu_n^{\otimes 2}) \to 0,$$

where $N^1H^3_{\text{ét}} \subset H^3_{\text{ét}}(X, \mu_n^{\otimes 2})$ is the subgroup of cohomology classes which die when restricted to some nonempty open $U \subset X$.

COROLLARY (2.4). *Assume $N^1H^3_{\text{ét}}(X, \mu_n^{\otimes 2})$ finite (e.g. k algebraically closed). Then*:

$$\mathrm{Coker}(R_{n,F}) \textit{ finite} \Rightarrow H^1(X, \mathscr{K}_2/n\mathscr{K}_2) \textit{ finite} \Rightarrow {}_nCH^2(X) \textit{ finite}.$$

For example, if ${}_2\mathrm{Br}(F)$ is generated by quaternion algebras (i.e. $\mathrm{Coker}(R_{2,F})=(0)$) we get ${}_2CH^2(X)$ finite.

These results follow from a study of the norm residue map of sheaves $R_{n,X}: \mathscr{K}_2/n\mathscr{K}_2 \to \mathscr{H}^2(\mu_n^{\otimes 2})$, where $\mathscr{H}^2(\mu_n^{\otimes 2})$ is the Zariski sheaf associated to the presheaf $U \mapsto H^2_{\text{ét}}(U, \mu_n^{\otimes 2})$.

COROLLARY (2.5). *Assume k is algebraically closed of characteristic $\neq 2$, and that X is a surface. Then $\Gamma(X, \mathscr{K}_2/2^r\mathscr{K}_2)$ is finite for any $r \geqq 1$.*

Indeed the function field $k(X)$ is C_2, so the assertion follows from (1.2) and (2.3).

3. **A cohomological criterion.** Let F be a field containing the group μ_{l^∞} of all lth power roots of 1 for some fixed prime $l \neq \text{char } F$. The diagram

$$\begin{array}{ccc} K_2(F)/l^\nu K_2(F) & \xrightarrow{R_{l^\nu,F}} & {}_{l^\nu}\mathrm{Br}(F) \otimes \mu_{l^\nu} \\ \downarrow \text{“}l\text{''} & & \downarrow \\ K_2(F)/l^{\nu+1}K_2(F) & \xrightarrow{R_{l^{\nu+1},F}} & {}_{l^{\nu+1}}\mathrm{Br}(F) \otimes \mu_{l^{\nu+1}} \end{array}$$

is commutative, and so taking inj $\lim_\nu$ we get

$$R_{l^\infty,F}: K_2(F) \otimes \boldsymbol{Q}_l/\boldsymbol{Z}_l \to \mathrm{Br}(F) \otimes \boldsymbol{Z}_l(1)$$

where $\boldsymbol{Z}_l = \text{proj } \lim_\nu \mu_{l^\nu}$.

THEOREM (3.1). *Let F_0 be a field with $\mu_{l^\infty} \subset F_0$. Assume* $\mathrm{Ker}(R_{l,F}) = (0)$ *for all F algebraic over F_0. Then* $\mathrm{Coker}(R_{l^\infty,F}) = (0)$ *for all such F if and only if the galois cohomology groups* $H^1(G, K_2'(F')) = (0)$ ($K_2' = K_2/\text{torsion}$) *whenever F'/F is a galois extension of fields algebraic over F_0, with*

$$G = \mathrm{Gal}(F'/F) \cong \boldsymbol{Z}/l\boldsymbol{Z}.$$

Combining this with (1.2) and (2.4), one gets a criterion for "cofiniteness" of the two torsion subgroup $CH^2(X)(2)$ of zero cycles on a surface.

ADDED IN PROOF. Lam and Elman **[14]** have proved independently of the author a sharper version of (1.2) which implies finiteness for $\Gamma(X, \mathscr{K}_2/2^r\mathscr{K}_2)$ when X has dimension $\leqq 3$ over an algebraically closed field of characteristic $\neq 2$ (cf. (2.5)).

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