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A WELL-POSED PROBLEM FOR THE HEAT EQUATION

BY THOMAS I. SEIDMAN

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ABSTRACT. Simultaneous specification of (consistent) Dirichlet and Neumann data boundedly determines later internal states of the solution of the heat equation in a general region.

We consider solutions of the heat equation $u_t = \Delta u$ for $0 < t < T$, $x = (x_1, \cdots, x_n) \in \Omega$. It is well known that arbitrary specification of both the *initial state* $u_0 = u(0, \cdot)$ and either *Dirichlet data*:

$$u(t, x) = f(t, x) \quad \text{for } 0 \leqq t \leqq T,\ x \in \partial\Omega,$$

or *Neumann data*:

$$(\partial u/\partial \nu)(t, x) = g(t, x) \quad \text{for } 0 \leqq t \leqq T,\ x \in \partial\Omega,$$

determines uniquely the evolution of the process. In particular, the *terminal state* $u_T = u(T, \cdot)$ is determined by either of the pairs (u_0, f), (u_0, g).

If the initial internal state is not given, we ask whether knowledge of *both* Dirichlet *and* Neumann data suffices. The pair (f, g) cannot be specified arbitrarily, but we adopt the viewpoint that in observation of an ongoing process, the consistency conditions are automatically satisfied so the observed pair (f, g) lies in the admissible manifold M, and the existence of a solution is not at issue. We ask whether *observation* of the boundary data (f, g) suffices for effective *prediction* of the terminal internal state u_T.

THEOREM. *The observation/prediction problem for the heat equation is well posed for any bounded region Ω in $\boldsymbol{R}^n$ with smooth boundary $\partial\Omega$. I.e., in the above notation, the map*: $(f, g) \mapsto u_T$ *is well defined and continuous, using appropriate $\mathscr{L}_2$ topologies for domain and range.*

SKETCH OF PROOF. (a) There is a reduction to a *restricted* problem of the same form with $g \equiv 0$.

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(b) Well-posedness of the restricted problem is equivalent to existence for each initial state in $\mathscr{L}_2(\Omega)$, of a Neumann null-control (i.e., of Neumann data g which, for the initial state u_0, gives $u_T \equiv 0$).

(c) Null-controllability for a region Ω_0 implies Neumann null-controllability, as above, for any Ω contained in Ω_0.

(d) Enclose Ω in an n-cube or n-ball Ω_0 for which—by results of [**2**] or [**3**], respectively—the restricted problem is well posed. By (b) this implies Neumann null-controllability for Ω_0 (alternatively, this is already given for the n-ball by results in [**1**]) and so, by (c), for Ω itself (alternatively, this is given by results in [**4**]). Then (b) implies well-posedness of the restricted observation/prediction problem for the heat equation in Ω and so, by the reduction (a), of the general problem. □

We remark that the steps (a), (b), (c) are valid for diffusion processes governed by parabolic equations with variable coefficients. The steps (a), (b) also apply to the interesting situation in which observation of the Dirichlet data is restricted to an open subset of $[0, T] \times \partial\Omega$; combining this with results in [**5**] gives well-posedness of that observation/prediction problem for the heat equation in star-complemented regions in $\boldsymbol{R}^n$. Details of the arguments above will appear elsewhere.

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Department of Mathematics, University of Maryland Baltimore County. Baltimore, Maryland 21228