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THE PERMANENT AT A MINIMUM ON CERTAIN CLASSES OF DOUBLY STOCHASTIC MATRICES[1]

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ABSTRACT. Both David London and Mark B. Hedrick have independently generalized a result of M. Marcus and M. Newman concerning the behavior of the permanent at a minimum on the set of doubly stochastic matrices. The author generalizes this last result and simplifies the proof appreciably. He proves the following. Let A be a doubly stochastic matrix and let X be a set of doubly stochastic matrices with the same (0, 1)-pattern as A in some neighborhood of A. If A is a critical point of the permanent relative to X, then per $A = \text{per}A(i|j)$ for each positive a_{ij}.

In 1926 [**8**], B. L. van der Waerden conjectured that the permanent achieves a unique minimum on the set D_n of $n \times n$ doubly stochastic matrices at the matrix J_n (each of whose entries is $1/n$). The conjecture has some interesting interpretations in finite probability and finite combinatorics. However, it is only known to be true for n less than or equal to 5 [**6**], [**1**], [**2**] and for the class of positive semidefinite, hermitian matrices [**5**]. The most general result previously known was obtained independently by D. London [**4**] and by M. B. Hedrick [**3**] and stated that if A is a matrix in D_n at which the permanent achieves a local minimum relative to D_n, then per $A \leqq \text{per } A(i|j)$ with equality for each positive a_{ij}. Both of these papers relied heavily on methods from the definitive work of M. Marcus and M. Newman [**6**] in which knowledge of the eigenvalues of AA^T was required. Since the relationship between the eigenvalues of A or AA^T and the permanent of A appears to be extremely nebulous [**7**], the author finds a great deal of beauty in the simplicity and purely combinatorial nature of the following proof.

THEOREM. *Let A be a doubly stochastic matrix, and let X be a set of doubly stochastic matrices with the same* (0, 1)*-pattern as A in some neighborhood of A. If A is a critical point of the permanent relative to X, then* per $A = \text{per } A(i|j)$ *for each positive a_{ij}.*

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[1] This paper is dedicated to Terrie.

PROOF. Assume that for some positive a_{ij}, per $A \neq$ per $A(i|j)$. Observe that for any fixed i, the permanent of A equals the sum of the a_{ij} per $A(i|j)$ when one sums on j for positive a_{ij}. Thus $\min_j$ per $A(i|j) \leqq$ per $A \leqq \max_j$ per$(i|j)$ with equality at either extreme if and only if there is equality throughout. A similar remark can be made for fixed j.

Let $a_{i(1),j(1)}$ be a positive entry in A at which the per $A(i(1)|j(1))$ equals max per $A(i|j)$ for all positive a_{ij}. By the previous remarks, per $A <$ per $A(i(1)|j(1))$. Likewise, there is some positive $a_{i(1),j(2)}$ such that per $A(i(1)|j(2))$ is less than per A. Then there is some positive $a_{i(2),j(2)}$ such that per $A(i(2)|j(2))$ is greater than per A. Within n times, we must return to a row or a column which has been used previously. For notational convenience, we shall assume that $a_{i(1),j(1)}$, $a_{i(1),j(2)}, \cdots, a_{i(p),j(p-1)}$, $a_{i(p),j(1)}$ are positive entries of A such that per $A(i(1)|j(1))$ is greater than per A, per $A(i(1)|j(2))$ is less than per $A, \cdots$, per $A(i(p)|j(p-1))$ is greater than per A, and per $A(i(p)|j(1))$ is less than per A, and the rows and columns which are used occur exactly twice.

Choose $0 < x < \min\{a_{i(1),j(1)}, \cdots, a_{i(p),j(1)}\}$. Consider the doubly stochastic matrix $A(x)$ defined by $a_{ij}(x) = a_{ij}$ for (i,j) different from $(i(1),j(1)), \cdots, (i(p),j(1))$ and

$$a_{i(1),j(1)}(x) = a_{i(1),j(1)} - x,$$
$$a_{i(1),j(2)}(x) = a_{i(1),j(2)} + x,$$
$$\vdots$$
$$a_{i(p),j(p-1)}(x) = a_{i(p),j(p-1)} - x,$$

and

$$a_{i(p),j(1)}(x) = a_{i(p),j(1)} + x.$$

Define $f(x)$ to be per $A(x)$. Then since $f(0)$ equals per A and A is a critical point of the permanent relative to X, the derivative of $f(x)$ evaluated at 0 equals 0. Thus

$$[\text{per } A(i(1)\,|\,j(2)) + \cdots + \text{per } A(i(p)\,|\,j(1))] - [\text{per } A(i(1)\,|\,j(1)) + \cdots + \text{per } A(i(p)\,|\,j(p-1))]$$

equals the derivative of $f(x)$ evaluated at 0 which equals 0. However, by the choice of the per $A(i(1)|j(1)), \cdots$, per $A(i(p)|j(1))$, the above sum is negative which is a contradiction.

COROLLARY I. *If A is a matrix at which the permanent achieves a local minimum relative to D_n, then* per $A \leqq$ per $A(i|j)$ *with equality whenever a_{ij} is positive.*

The procedure to prove Corollary I can be found in [**3**].

COROLLARY II. *If A is a matrix at which the permanent achieves a local maximum relative to D_n, then* per $A \geqq$ per $A(i|j)$ *with equality whenever a_{ij} is positive.*

Write A as a direct sum of fully indecomposable matrices and use the techniques in [3].

It is worth noting that the boundary of D_n satisfies the hypothesis of the Theorem. Likewise, let B be a doubly stochastic matrix. Define X to be the set of all doubly stochastic matrices C such that c_{ij} is zero whenever b_{ij} is zero. Then X satisfies the hypothesis of the Theorem.

M. Marcus and M. Newman showed that the permanent achieves a local minimum at J_n [6], and it is immediate that the permanent achieves an absolute maximum at a permutation matrix (take the product of the n row sums). The author would be very interested in seeing examples at which the permanent achieves a local minimum or local maximum relative to D_n besides these.

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