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EXTREMAL LENGTH, REPRODUCING DIFFERENTIALS AND ABEL'S THEOREM

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Let c be a 1-chain on a Riemann surface R and $\Gamma_x(R)$ a closed subspace of $\Gamma_h(R)$, the Hilbert space of square integrable harmonic differential forms on R, then there is a unique $\psi_x(c) \in \Gamma_x(R)$ such that $\int_c \omega = (\omega, \psi_x(c))$ for all $\omega \in \Gamma_x(R)$. $\psi_x(c)$ is called the $\Gamma_x(R)$-reproducing differential for c and $\|\psi_x(c)\|^2$ is a conformal invariant. For the case of a 1-cycle c an extremal length interpretation for the squared norm of the reproducing differential was given by Accola [1] and Blatter [2] for $\Gamma_h(R)$, by Marden [3] for $\Gamma_{ho}(R)$ and by Rodin [5] for $\Gamma_{hse}(R)$. In each of these results the curve family whose extremal length gave the square of the norm of the reproducing differential was a homology class associated with c. Rodin [5] asked whether there were similar theorems for other subspaces of $\Gamma_h(R)$ and what the proper curve family would be in case c was an arbitrary 1-chain, not necessarily a 1-cycle. If c is a single arc, then a reduced extremal distance interpretation of the norm of the reproducing differential for $\Gamma_{he}(R)$, $\Gamma_{hm}(R)$ and $\Gamma_{he}(R) \cap \Gamma^*_{hse}(R)$ was given in [4]. The purpose of this paper is to announce solutions to the problems posed by Rodin for a large number of important subspaces of $\Gamma_h(R)$; a complete, detailed paper is forthcoming.

For the sake of simplicity we shall consider only compact Riemann surfaces; this case gives rise to one of the most important applications. Let c be a 1-chain on the compact Riemann surface R. Suppose that $\partial c = \sum_{j=1}^{J} n_j b_j - \sum_{i=1}^{I} m_i a_i$, where the points a_i, b_j are all distinct and m_i, n_j are positive integers, unless $\partial c = 0$. Define $\mathcal{F} = \mathcal{F}(c) = \{d : d$ is a 1-chain on R and $\partial d = \partial c\}$ and $\mathcal{H} = \mathcal{H}(c) = \{d : d \in \mathcal{F}$ and $c - d$ is homologous to $0\}$. Consider fixed local coordinates w_i, z_j defined in a neighborhood of a_i, b_j respectively. Given vectors $\boldsymbol{r} = (r_1, \cdots, r_I)$ and $\boldsymbol{s} = (s_1, \cdots, s_J)$ of positive numbers, let $R(\boldsymbol{r}, \boldsymbol{s})$ be the bordered Riemann surface obtained by removing from R disks of radius r_i, s_j about a_i, b_j,

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relative to these local coordinates. Set $\mathscr{F}(r, s)=\{\mathscr{d}\cap R(r, s):\mathscr{d}\in\mathscr{F}\}$ and

$$\tilde{\lambda}(\mathscr{F}) = \lim_{r,s\downarrow 0} \lambda(\mathscr{F}(r, s)) + \frac{1}{2\pi}\left(\sum_{i=1}^{I} m_i^2 \log r_i + \sum_{j=1}^{J} n_j^2 \log s_j\right).$$

$\tilde{\lambda}(\mathscr{F})$ exists and is called the reduced extremal length of the family $\mathscr{F}$ with respect to the local coordinates w_i, z_j. This quantity depends upon the choice of local coordinates in such a way that

$$\exp(-2\pi\tilde{\lambda}(\mathscr{F})) \prod |dw_i|^{m_i^2} \prod |dz_j|^{n_j^2}$$

is an invariant form. $\tilde{\lambda}(\mathscr{H})$ is defined in a similar fashion.

We also associate two singular differentials with c. Let p be a harmonic function on R such that in a neighborhood of a_i, $p=(m_i/2\pi)\log|w_i-a_i|+u_i$, where u_i is harmonic at a_i, and near b_j, $p=-(n_j/2\pi)\log|z_j-b_j|+v_j$, where v_j is harmonic at b_j. The function p exists and is determined up to an additive constant. Set $\tilde{\psi}_0=\tilde{\psi}_0(c)=dp$ and $\tilde{\psi}_h=\tilde{\psi}_h(c)=\psi_h-\tilde{\psi}_0$, then for any $\omega \in \Gamma_h(R)$, $0=(\omega, \tilde{\psi}_0)$ and $\int_c \omega=(\omega, \tilde{\psi}_h)$. These inner products both exist since the integrals which give the inner products converge absolutely even though $\tilde{\psi}_0$ and $\tilde{\psi}_h$ have singularities. Set

$$\langle\langle\tilde{\psi}_h\rangle\rangle^2 = \lim_{r,s\downarrow 0} \|\tilde{\psi}_h\|^2_{R(r,s)} + \frac{1}{2\pi}\left(\sum_{i=1}^{I} m_i^2 \log r_i + \sum_{j=1}^{J} n_j^2 \log s_j\right).$$

This quantity exists but is not invariantly defined; however,

$$\exp(-2\pi\langle\langle\tilde{\psi}_h\rangle\rangle^2) \prod |dw_i|^{m_i^2} \prod |dz_j|^{n_j^2}$$

is an invariant form. $\langle\langle\tilde{\psi}_0\rangle\rangle^2$ is defined analogously. It can be shown that $\langle\langle\tilde{\psi}_h\rangle\rangle^2-\langle\langle\tilde{\psi}_0\rangle\rangle^2=\|\psi_h\|^2$.

The following theorem is our main result.

THEOREM. *$\tilde{\lambda}(\mathscr{F}(c))=\langle\langle\tilde{\psi}_0(c)\rangle\rangle^2$ and $\tilde{\lambda}(\mathscr{H}(c))=\langle\langle\tilde{\psi}_h(c)\rangle\rangle^2$.*

COROLLARY. $\|\psi_h(c)\|^2=\tilde{\lambda}(\mathscr{H}(c))-\tilde{\lambda}(\mathscr{F}(c))$.

This corollary leads to an extremal length interpretation of Abel's theorem. Let D be a divisor on the compact Riemann surface R. Assume that either $D=0$ or $D=B-A$, where A and B are disjoint integral divisors; that is, $A=\sum_{i=1}^{I} m_i a_i$ and $B=\sum_{j=1}^{J} n_j b_j$, the points a_i, b_j all being distinct and m_i, n_j being positive integers. D is called a principal divisor if there is a rational function f on R such that the divisor of f is D. Abel's theorem asserts that D is a principal divisor if and only if there is a 1-chain c on R with the property that $\partial c=D$ and $\int_c \omega=0$ for all $\omega \in \Gamma_h(R)$. Now, in order that $\int_c \omega=0$ holds for all $\omega \in \Gamma_h(R)$, it is necessary and sufficient that $\|\psi_h(c)\|=0$. Consequently, the next theorem has been established.

THEOREM. *A divisor D on a compact Riemann surface R is principal if and only if there is a 1-chain c on R with $\partial c = D$ and $\tilde{\lambda}(\mathscr{F}(c)) = \tilde{\lambda}(\mathscr{H}(c))$.*

Our main theorem has several analogs on an open Riemann surface. In fact, on an open surface there are six curve families associated with a 1-chain c. The reduced extremal length of all six families can be expressed in terms of singular differentials which are closely related to various reproducing differentials connected with c. By making use of these results, we can give an extremal length interpretation for the squared norm of the $\Gamma_x(R) \cap \Gamma_y^*(R)$-reproducing differential for a 1-chain c; here x and y can represent any one of h, hse, ho, he, hm, except that $x=y=he$ or ho is not permitted. There are fourteen nontrivial such subspaces of $\Gamma_h(R)$.

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