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BILINEAR FORMS AND CYCLIC GROUP ACTIONS

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In a recent paper [2] Conner and Raymond have given an approach to the study of smooth cyclic group actions which employs rational bilinear forms. If $K^{4n-1}=\partial B^{4n}$ bounds a compact oriented smooth manifold, there is a symmetric nonsingular rational bilinear form on the image of $H^{2n}(B, K; Q)\to H^{2n}(B; Q)$ which represents an element $w(B)$ in $W(Q)$, the rational Witt ring. Denoting the signature of this form by $\operatorname{sgn}(B)$ and the unit of $W(Q)$ by $\mathbf{1}$, the *peripheral invariant* of K,

$$\operatorname{per}(K) = w(B) - \operatorname{sgn}(B) \cdot \mathbf{1},$$

lies in the kernel of the signature homomorphism $\Phi: W(Q)\to Z$ and is independent of the choice of B. In [2] there is associated with any orientation preserving diffeomorphism (T, M^{4n}) of prime period p on a closed manifold an element of the kernel of Φ which we denote by $q(T, M)$, an invariant of the equivariant bordism class which vanishes on fixed point free actions. Using the peripheral invariant, Conner and Raymond computed $q(T, M)$, for $p=2$ or 3, in terms of the fixed point information. The fundamental problem posed in [2] is the extension of this result to all primes.

In this paper we give the general formula for all primes and apply it to establish relationships between the index of M and the index of the fixed set. The essence of the proof is a group isomorphism between the kernel of Φ and $\bigoplus_p W(Z_p)$ where $W(Z_p)$ is the Witt group of the field Z_p and the sum ranges over all primes. Using this isomorphism, we establish a relation between the peripheral invariant and the linking form which enables us to extend the definition of $\operatorname{per}(K)$ to any closed oriented $(4k-1)$-manifold.

1. **Bilinear forms.** Let $B\,\mathrm{Fin}$ denote the semigroup of isomorphism classes of symmetric nonsingular bilinear forms on finite abelian groups taking values in Q/Z. Denote by $W^s(Z)$ the semigroup of stable equivalence classes of nondegenerate integral bilinear forms on finitely

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generated free abelian groups, where stability means that we are allowed to add the form x^2 or the form $-x^2$ without altering the equivalence class. Let $W^s(\boldsymbol{Q})$ be the corresponding group of rational bilinear forms. Kneser and Puppe [**5**] have shown that there is a one-to-one correspondence between $W^s(\boldsymbol{Z})$ and B Fin (see also [**3**], [**6**], [**7**], [**8**]).

The composition $B\text{ Fin}\to W^s(\boldsymbol{Z})\to W^s(\boldsymbol{Q})$ is clearly onto and generates an equivalence relation $\sim$ on B Fin, which we refer to as rational equivalence of finite forms. Suppose (λ, G) is a finite form in B Fin and $K\subseteq H\subseteq G$ are subgroups such that

$$H = \{x \in G \mid \lambda(x, y) = 0 \quad \text{for all } y \in K\} = K^{\perp},$$

the annihilator of K. λ induces a nonsingular form λ' on H/K.

1.1. THEOREM. *If G, H, K, λ and λ' are as given above, $(\lambda, G)\sim(\lambda', H/K)$ in B* Fin. *Conversely, if (λ, G) is rationally trivial, there is a subgroup $H\subseteq G$ such that $H=H^{\perp}$.*

We denote by $\mathscr{W}$ the Grothendieck group generated by B Fin modulo the subgroup generated by all forms (λ, G) such that there is a subgroup $H\subseteq G$ with $|H|^2=|G|$ and $\lambda(H, H)=0$.

1.2. THEOREM. *If $W(\boldsymbol{Z}_p)$ denotes the Witt group of nonsingular bilinear forms over $\boldsymbol{Z}_p$, the inclusion induces an isomorphism of groups, $\bigoplus_p W(\boldsymbol{Z}_p)\xrightarrow{\approx}\mathscr{W}$, where the sum ranges over all primes p.*

We can summarize the above results in the following corollary [**9**].

1.3. COROLLARY. *There is a sequence of group isomorphisms*

$$W^s(\boldsymbol{Q}) \approx \mathscr{W} \approx \bigoplus_p W(\boldsymbol{Z}_p),$$

and since $W^s(\boldsymbol{Q})$ may be identified with the kernel of the signature homomorphism $\Phi: W(\boldsymbol{Q})\to(\boldsymbol{Z})$, there is an isomorphism of groups (but not of rings)

$$W(\boldsymbol{Q}) \approx W(\boldsymbol{R}) \oplus \left(\bigoplus_p W(\boldsymbol{Z}_p)\right).$$

A form $\lambda:\boldsymbol{Z}_p\times\boldsymbol{Z}_p\to\boldsymbol{Q}/\boldsymbol{Z}$ with $\lambda(1, 1)=b/p$ is completely determined up to isomorphism by $[b]\in\boldsymbol{Z}_p^*/\boldsymbol{Z}_p^{**}$, the multiplicative group of units modulo squares. Denote the form $\lambda(1, 1)=b/p$ by $\langle b\rangle_p$. As abelian groups we have for $p\equiv 3 \pmod 4$, $W(\boldsymbol{Z}_p)\approx\boldsymbol{Z}_4$ generated by $\langle 1\rangle_p$; for $p\equiv 1 \pmod 4$, $W(\boldsymbol{Z}_p)\approx\boldsymbol{Z}_2\oplus\boldsymbol{Z}_2$ generated by $\langle 1\rangle_p$ and $\langle a\rangle_p$ where a is not a square mod p, and $W(\boldsymbol{Z}_2)\approx\boldsymbol{Z}_2$ generated by $\langle 1\rangle_2$. The integral form corresponding to $\langle 1\rangle_p$ is the 1×1 matrix (p). There is a concise algorithm for constructing the matrix for the integral form corresponding to $\langle a\rangle_p$.

2. **Prime order actions.** Let M^{4n-1} be a closed oriented smooth manifold and let G be the torsion subgroup of $H^{2n}(M; \boldsymbol{Z})$. Recall the definition of the linking form λ on M: If $x, y \in G$ then $x=\beta(z)$ for some $z \in H^{2n-1}(M; \boldsymbol{Q}/\boldsymbol{Z})$ where β is the Bockstein homomorphism. Then $\lambda(x, y)= \langle z \cup y, [M]\rangle$ in $\boldsymbol{Q}/\boldsymbol{Z}$ defines a nonsingular bilinear pairing on G. Denote the corresponding element of $\mathscr{W}$ by $\lambda(M)$.

Suppose that $M^{4n-1}=\partial B^{4n}$ where B^{4n} is a compact oriented smooth manifold. For $i:M\to B$ the inclusion map, define

$$H = \{x \in G \mid x = i^*(\xi) \text{ for some } \xi \in H^{2n}(B; \boldsymbol{Z})\}, \text{ and}$$
$$K = \{x \in G \mid x = i^*(\xi) \text{ for some } \xi \in \operatorname{Tor} H^{2n}(B; \boldsymbol{Z})\}.$$

Then K is isomorphic to G/H. Note that a necessary condition for B^{4n} to be a rational disk is that $|G|$ be a square, since in this case $H=K$. This gives information on a question posed in [**1**].

2.1. COROLLARY. *If T is a smooth diffeomorphism of prime period on S^{2k-1} with fixed set M^{4n-1} such that the order of* $\operatorname{Tor}(H^{2n}(M; \boldsymbol{Z}))$ *is not a square, then T cannot be smoothly extended to D^{2k}.*

2.2. LEMMA. *Under the linking form (λ, G), $H=\{x \in G \mid \lambda(x, y)= 0$ for all $y \in K\}=K^{\perp}$.*

2.3. THEOREM. *If $M^{4n-1}=\partial B^{4n}$ as above, then under the isomorphism $W^s(\boldsymbol{Q})\approx\mathscr{W}$,*

$$\operatorname{per}(M) = -\lambda(M).$$

By taking this equation as the definition, the peripheral invariant may be extended to all closed oriented $(4n-1)$-manifolds (in fact, using techniques analogous to those for the index, it can be extended to compact manifolds with boundary).

2.4. COROLLARY. $\operatorname{per}(M)$ *is defined for all closed oriented* $(4n-1)$-*manifolds and is an invariant of the oriented homotopy type of M.*

2.5. COROLLARY. *If T is a smooth diffeomorphism of prime period on S^{2k-1} with fixed set M^{4n-1} having* $\operatorname{per}(M)\neq 0$, *then T cannot be smoothly extended to D^{2k}.*

The lens spaces give an interesting set of examples for examining the peripheral invariant as well as for applications. The quotient of the action of $\boldsymbol{Z}_p$ on S^{4n-1} given by $T(z_1, \cdots, z_{2n})=(\alpha^{r_1}z_1, \cdots, \alpha^{r_{2n}}z_{2n})$, where $\alpha=e^{2\pi i/p}$ and $(r_j, p)=1$, gives the lens space $L^{4n-1}(p; r_1, \cdots, r_{2n})$. For each j choose an integer l_j with $l_j \cdot r_j=1 \bmod p$. Let $l=l_1 \cdot l_2 \cdot \cdots \cdot l_{2n}$.

2.6. PROPOSITION. *The linking form on this lens space is given by $\lambda(L^{4n-1}(p; r_1, \cdots, r_{2n}))=\langle l\rangle_p$.*

Conner and Raymond [2] have defined an invariant for smooth periodic group actions that fits nicely into this setting. Let (T, M^{4n}) be an orientation preserving diffeomorphism of odd prime period p on a closed manifold. There is a symmetric, nonsingular bilinear form on $H^{2n}(M; \mathbf{Q})$ given by $f(x, y)=p \cdot \langle x \cup y, [M]\rangle \in \mathbf{Q}$. The restriction of f to the fixed vectors defines an element $w(T, M) \in W(\mathbf{Q})$, whose signature we write as $\operatorname{sgn}(M/T)$. The invariant defined in [2] which we have denoted by $q(T, M)$ is defined by

$$q(T, M) = w(T, M) - \operatorname{sgn}(M/T) \cdot \mathbf{1}.$$

Since this lies in the kernel of $\Phi: W(\mathbf{Q}) \rightarrow \mathbf{Z}$, we view it as an element of $\mathscr{W}$. One of the principal results of [2] is the determination of this invariant for $p=2$ or 3.

2.7. THEOREM (CONNER AND RAYMOND [2]). *If $p=3$, or if $p=2$ and T is weakly complex, then $q(T, M)=\operatorname{sgn}(F) \cdot \langle 1\rangle_p$ where F is the fixed set.*

Let N be an equivariant tubular neighborhood of F in M. The relationship between this invariant and the peripheral invariant may be stated [2] as

$$q(T, M) = p \otimes w(N) - \operatorname{sgn}(N) \cdot \mathbf{1} - \operatorname{per}(\partial N/T)$$

where tensoring a rational form with p corresponds to multiplying each entry in its matrix by p.

Now suppose that F_0^{2k} is a component of the fixed set and $S^{2m-1} \rightarrow \partial N_0 \rightarrow F_0^{2k}$ is the equivariant sphere bundle over F_0. The quotient under T is the lens space bundle $L_0^{2m-1} \rightarrow \partial N_0/T \rightarrow F_0^{2k}$.

An argument involving spectral sequences shows:

2.8. THEOREM. *The local fixed point information is given by*

$$p \otimes w(N_0) - \operatorname{sgn}(N_0) \cdot \mathbf{1} - \operatorname{per}(\partial N_0/T) = \operatorname{sgn}(F_0^{2k}) \cdot \lambda(L_0^{2m-1}).$$

2.9. COROLLARY. (a) *For $p \equiv 3 \pmod 4$ give the normal bundle to F a complex structure in which all eigenvalues are of the form α^k where k is a square* mod p. *Then if F is given the orientation consistent with the orientation of M,*

$$q(T, M) = \operatorname{sgn}(F) \cdot \langle 1\rangle_p.$$

(b) *For $p \equiv 1 \pmod 4$ orient F arbitrarily. Let F_1 be the union of those components of F in which the corresponding lens space has $\lambda(L)=\langle 1\rangle_p$ and F_2 the union of the remaining components. Then*

$$q(T, M) = \operatorname{sgn}(F_1) \cdot \langle 1\rangle_p + \operatorname{sgn}(F_2) \cdot \langle a\rangle_p.$$

2.10. COROLLARY. *Suppose* (T, M) *is as above and* T^* *is the identity on* $H^{2n}(M; Q)$. *Then with* F *oriented as in* (2.9),

(a) *For* $p \equiv 3 \pmod 4$, $\operatorname{sgn}(M) \equiv \operatorname{sgn}(F) \bmod 4$.

(b) *For* $p \equiv 1 \pmod 4$, $\operatorname{sgn}(M) \equiv \operatorname{sgn}(F) \equiv \operatorname{sgn}(F_1) \bmod 2$.

(Note that a unimodular form of dimension less than p has no isometries of order p.)

Related results comparing the index of M to the index of F have been obtained by Lowell Jones [**4**] using completely different methods. It may be seen by simple examples that the relation in (a) is the best possible. We have evidence that in (b) there may be a Z_4 invariant. In fact for $p=5$ an argument using the Atiyah-Singer index theorem shows that $\operatorname{sgn}(M) \equiv \operatorname{sgn}(F) \bmod 4$.

REFERENCES

1. J. P. Alexander, G. C. Hamrick and J. W. Vick, *Involutions on homotopy spheres*, (1973) (preprint).

2. P. E. Conner and F. Raymond, *A quadratic form in the quotient of a periodic map*, (1973) (preprint).

3. A. Durfee, Ph.D. Thesis, Cornell University, Ithaca, N.Y., 1971.

4. L. Jones, *Combinatorial symmetries of the disk* (preprint).

5. M. Kneser and D. Puppe, *Quadratische Formen und Verschlingungsinvarianten von Knoten*, Math. Z. **58** (1953), 376–384. MR **15**, 100.

6. C. T. C. Wall, *Quadratic forms on finite groups, and related topics*, Topology **2** (1964), 281–298. MR **28** #133.

7. ———, *Quadratic forms on finite groups*. II, Bull. London Math. Soc. **4** (1972), 156–160.

8. Wilkins, Ph.D. Thesis, University of Liverpool, 1971.

9. D. Husemoller and J. W. Milnor, *Symmetric bilinear forms*, Ergebnisse der Mathematik und ihrer Grenzgebiete, **73** (1973).

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