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SETS OF COLORINGS OF CIRCUITS

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1. **Introduction.** A circuit Γ is a triangulation of the one-dimensional sphere S^1. It shall have as its set of vertices $\Gamma_0 = Z_k = \{0, 1, \cdots, k-1\}$, and as its set of one-simplices $\Gamma_1 = \{\sigma_j = (j-1, j) | j = 1, 2, \cdots, k\}$. A coloring of Γ is a zero-dimensional cochain $c^0 \in C^0(\Gamma, Z_2 \oplus Z_2)$ whose coboundary is "nowhere zero", i.e. $\delta c^0(\sigma_j) \neq 0$ for all $\sigma_j \in \Gamma_1$. A set K of colorings of Γ is realizable as a set of admissible colorings if there is a triangulated two-dimensional disk D with boundary Γ such that the restriction homomorphism

$$j^{\#}: C^0(D, Z_2 \oplus Z_2) \to C^0(\Gamma, Z_2 \oplus Z_2)$$

(induced by the inclusion $j: \Gamma \to D$) takes the colorings of D onto K.

Let $\psi(k)$ be the minimum cardinality of a set K which is realizable as a set of admissible colorings.

REMARK 1. $\psi(k) = 0$ if and only if the four color conjecture is false.

The conjecture of Albertson and Wilf [1]. $\psi(k) = 3 \cdot 2^k$ for $k = 3, 4, \cdots$.

Comment 1. Since $3 \cdot 2^k$ is the number of colorings of any disk D with no interior vertices and k vertices in $\Gamma_0 = D_0$, we conclude $3 \cdot 2^k \geqq \psi(k)$.

Comment 2. It is not known whether the four color conjecture implies the Albertson-Wilf conjecture for $k > 6$. (It does for $k = 3, 4, 5$ and 6 [1].)

In [1], Albertson and Wilf announce:

THEOREM 1. *If the four color conjecture holds then*

$$\psi(k) \geqq (4!) F_{k-1} \geqq C((1 + \sqrt{5})/2)^k$$

where F_k is the kth Fibonacci number.

By generalizing the notion of a set of admissible colorings of Γ to the notion of a complete set of colorings of Γ, one can prove by induction on k:

THEOREM 2. *If the four color conjecture holds then*

$$\psi(k) > 4 \cdot 3^{k/2} \quad \textit{if } k \textit{ is even,}$$
$$> 8 \cdot 3^{(k-1)/2} \quad \textit{if } k \textit{ is odd.}$$

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2. **Coboundaries of colorings.** Let D be a triangulated two-dimensional disk with boundary Γ. Since D is connected, $H^0(D, Z_2 \oplus Z_2) \approx Z_2 \oplus Z_2$. Hence there are exactly four colorings corresponding to each nowhere zero one-dimensional cobounding cocycle. All of the sets of colorings that we consider will contain all four colorings with a given coboundary if the set contains any one of them. Thus we can consider the sets of coboundaries of colorings as easily as the sets of colorings.

The disk D is contractible so $H^1(D, Z_2 \oplus Z_2) \approx 0$. Hence the group of cocycles $Z^1(D, Z_2 \oplus Z_2)$ is equal to the group of cobounding cocycles $B^1(D, Z_2 \oplus Z_2)$. [$H^1(\Gamma, Z_2 \oplus Z_2) \approx Z_2 \oplus Z_2$. In this case the cobounding cocycles are characterized by the sum of all values being zero.]

Notation. $Z_2 \oplus Z_2 = \{0, e_1, e_2, e_3\}$ with the obvious addition.

If z is a nowhere zero cocycle on D and τ^2 is a two-simplex with faces α, β and γ, then $z(\alpha)+z(\beta)+z(\gamma)=0$. Hence z assigns the three values e_1, e_2 and e_3 to the faces α, β and γ of τ^2. Let us suppose that $z(\beta)=e_3$. We may hold the value e_3 fixed and interchange the values e_1 and e_2 on α and γ. This change will propagate along a Z_2 cocycle which contains either no one-simplexes from Γ_1 or exactly two one-simplexes from Γ_1.

REMARK 2. Let z be a nowhere zero cocycle on D and e_i a fixed value in $Z_2 \oplus Z_2$. For each σ_j in Γ_1 with $z(\sigma_j) \neq e_i$, there is a uniquely determined $\sigma_{j'} \in \Gamma_1$ and z' a nowhere zero cocycle on D such that:

(i) $j' \neq j$.

(ii) For every $\alpha \in D_1$, $z'(\alpha)=e_i$ if and only if $z(\alpha)=e_i$.

(iii) For every $\sigma_l \in \Gamma_1$

$$\begin{aligned} z'(\sigma_l) &= z(\sigma_l) + e_i \quad \text{if } l = j, j', \\ &= z(\sigma_l) \qquad\quad \text{otherwise.} \end{aligned}$$

The pairing $\sigma_j \leftrightarrow \sigma_{j'}$ is called a planar change diagram for $j^{\#}(z)$ and e_i. [$j^{\#}(z) \in Z^1(\Gamma, Z_2 \oplus Z_2)$.] It satisfies:

(i) If $\sigma_j \leftrightarrow \sigma_{j'}$, then neither $z(\sigma_j)=e_i$ nor $z(\sigma_{j'})=e_i$. Furthermore if $z(\sigma_j) \neq e_i$ then σ_j belongs to a pair.

(ii) If $\sigma_j \leftrightarrow \sigma_{j'}$ and $\sigma_l \leftrightarrow \sigma_{l'}$ then σ_l and $\sigma_{l'}$ lie on the same arc between σ_j and $\sigma_{j'}$.

Let z be a nowhere zero cobounding cocycle on Γ and let P be a planar change diagram for z and e_i. With each set of pairs of P we can associate a nowhere zero cobounding cocycle z' on Γ. This association is called the action of P on z. If z is in some set L of cocycles and $z' \in L$ for all sets of pairs of P then we say L is closed under the action of P.

DEFINITION. A complete set K of colorings of Γ corresponds to a set δK of nowhere zero cobounding cocycles with the properties:

(i) K is invariant under the action of the six automorphisms $\nu: Z_2 \oplus Z_2 \rightarrow Z_2 \oplus Z_2$.

(ii) For each $z \in \delta K$ and e_i there is a planar change diagram P so that δK is closed under the action of P.

3. **Induced sets.** A nondegenerate simplicial map $f: E \to F$ induces a homomorphism: $f^{\#}: B^1(F, Z_2 \oplus Z_2) \to B^1(E, Z_2 \oplus Z_2)$, which preserves the property of being nowhere zero. In general, however, complete sets of colorings on circuits are not preserved. [Let $f: \Gamma' \to \Gamma$ be a two-fold covering.]

4. **Potted trees.** A potted tree is a contractible simplicial complex with no more than one two-dimensional simplex. A circuit Γ is properly mapped to a potted tree T if $f: \Gamma \to T$ is a nondegenerate simplicial map such that for each $\alpha \in T_1$, $f^{-1}(\alpha)$ has exactly one or exactly two elements depending upon whether α is the face of a two simplex or not.

REMARK 3. If Γ is properly mapped to the potted tree T then T_1 has $k/2$ or $(k+3)/2$ elements. The set L of all colorings of T has $n(k)$ elements and $f^{\#}(L)$ is a complete set of colorings of Γ of cardinality $n(k)$ where

$$\begin{aligned} n(k) &= 4 \cdot 3^{k/2} \quad \text{if } k \text{ is even,} \\ &= 8 \cdot 3^{(k-1)/2} \quad \text{if } k \text{ is odd.} \end{aligned}$$

Comment 3. $n(k) \leqq 2 \cdot n(k-1)$ [equality when k is odd]; $n(k) = 3 \cdot n(k-2)$.

A set K of colorings of Γ is realizable as induced by a potted tree if there exists a proper map $f: \Gamma \to T$ such that $K = f^{\#}(L)$.

5. **Outline of the proof of Theorem 2.** From a circuit Γ, we can form a circuit Γ' by deleting the open star of the vertex 1 and by inserting a one-simplex (0, 2). We can also form a circuit Γ'' by performing the same deletion and identifying the vertices 0 and 2. Since every nowhere zero cobounding cocycle z on Γ induces either a cocycle z' on Γ' or a cocycle z'' on Γ'', a complete set of colorings K on Γ induces complete sets K' and K'' on Γ' and Γ'', respectively.

We prove by induction on the number k of vertices of Γ, that K has fewer than $n(k)$ elements only if K is empty. This follows from two inequalities. First the number of elements in K'' is less than or equal to one third the number of elements in K. Secondly, if K'' is empty then the number of elements in K' is less than or equal to half the number of elements in K. In essentially the same way we can prove that if K has exactly $n(k)$ elements then K can be realized as induced by a potted tree.

REFERENCE

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