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ON EXTENDED STRUCTURES OF A CLOSED OPERATOR RELATED TO SEMIGROUP THEORY AND THE ABSTRACT CAUCHY PROBLEM

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1. After the fundamental treatise on the semigroup theory by Hille and Phillips, several extensions of the semigroup theory and the abstract Cauchy problem have been developed. Three of the basic developments which are related to the results in this announcement are the distributional semigroups (V. Barbu [**1**], J. Chazarain [**4**], I. Ciorănescu [**5**], G. Da Prato and U. Mosco [**6**], H. O. Fattorini [**9**], C. Foiaş [**10**], D. Fujiwara [**11**], E. Larsson [**16**], J. L. Lions [**17**], J. Peetre [**19**], L. Schwartz [**21**], Ushijima [**23**], K. Yoshinaga [**24**]), the linear differential equations in Banach space (S. G. Kreĭn [**15**] contains a bibliography up to 1966, recently by R. Beals [**2**], [**3**], G. Da Prato and Giusti [**7**], M. Sova [**22**]), and the semigroup theory on a locally convex space (H. Komatsu [**13**], Kômura [**14**], I. Miyadera [**18**], K. Yosida [**25**]).

In this communication, we restrict ourselves to consider only a single closed operator A on a Banach space H in order to simplify the statements of our theorems. The problem is the abstract Cauchy problem (ACP), i.e., to find a solution in H for the differential equation $y'(t)=Ay(t)$ on the interval $[0, \infty)$ with $y(0)=x$ for some $x \in H$. This problem by various authors (for example, Hille and Phillips [**12**], Kreĭn [**15**]) is to find a strongly continuous semigroup $T(t)$ on $[0, \infty)$ of continuous operators on H such that $T(t)x$ is the solution of the problem for certain $x \in H$. In 1960, J. Lions proposed the study of the distributional semigroup which gives a distributional solution for the problem $y'(t)-Ay(t)=f(t)$. This extends the solvability of the ACP to a larger class of closed operators. What we are trying to do here is to give a formulation in between the two mentioned above. Formally, the idea is similar to that from the theory of partial differential equations, in the sense that the differential operator is solvable in an extended space which is the completion of the underlying space of the differential operator under a weaker topology.

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An indication of the proof for each theorem stated will be given, and most of the proofs can be obtained by applying the three basic principles in Dunford and Schwarz [**8**].

2. We will assume throughout this paper that A is a closed operator on a Banach space H with domain $D(A)$ dense in H, and the resolvent set $\rho(A)\neq\varnothing$. As usual H^* and A^* will be the adjoint of H and A respectively. Let us use the notation $D_H(A^n)_s$ to denote the space $D_H(A^n)$ with the Banach norm $\sum_{k=0}^{\infty}\|A^k x\|$. It follows that every element x in H can be naturally identified as an element in $D(A^*)_s^*$. Let H_e be the closure of H in $D(A^*)_s^*$. Then A as a linear transformation from $D(A)$ to H_e can be extended as a continuous linear transformation A_e on H. Furthermore, A_e is a closed operator on the Banach space H_e with domain $D_{H_e}(A_e)=H$ dense in H_e, and the resolvent $\rho_{H_e}(A_e)\neq\varnothing$. Define $A_1=A_e$ and $H_1=H_e$. Inductively, we can define $A_{n+1}=(A_n)_e$ and $H_{n+1}=(H_n)_e$. On the other hand, denote A_r to be the restriction of A on $D(A^2)$. Then A_r is a closed operator on $H_r=D(A)_s$ with domain $D_{H_r}(A_r)=D(A^2)$ dense in H_r, and $\rho_{H_r}(A_r)=\varnothing$. Define $H_{-1}=H_r$ and $A_{-1}=A_r$. Inductively, define for each negative integer n, $H_{n-1}=(H_n)_r$ and $A_{n-1}=(A_n)_r$. We summarize the properties of H_n and A_n in the following.

THEOREM 1. *There exists a sequence of Banach spaces H_n and a sequence of linear transformations A_r for each integer n such that*

(i) *$H_0=H$ and $A_0=A$;*
(ii) *H_n is contained and dense in H_{n+1};*
(iii) *A_{n+1} is a closed operator on H_{n+1} with domain $D_{H_{n+1}}(A_{n+1})=H_n$;*
(iv) *$H_{n+1}=(H_n)_e$ and $A_{n+1}=(A_n)_e$;*
(v) *conditions* (i) *to* (iv) *characterize the sequences H_n and A_n uniquely;*
(vi) *$H_{n-1}=(H_n)_r$ and $A_{n-1}=(A_n)_r$;*
(vii) $\rho_{H_n}=\rho_H(A)$.

3. Let n be a positive integer or $n=\infty$, $0\leqq b\leqq\infty$, and $x\in H$. We said that the abstract Cauchy problem of order n at the element x on the interval $[0, b]$ (or $[0, \infty)$ if $b=\infty$) for A on H abbreviated by ACP(n, x, b, A, H), is solvable if there exists an n times continuously differentiable function $y(t)=y(t, x)$ on the interval $[0, b]$ in H with $y(0, x)=x$ and $y^{(k)}(t)=A^k y(t)$ for all integers $0\leqq k<n+1$. By ACP$^*(n, x^*, b, A^*, H^*)$ is solvable, we mean the same as above except that continuously differentiable is replaced by strongly continuously differentiable. Define

$$X(n, b) = \{x\in H:\text{ACP}(n, x, b, A, H) \text{ is solvable}\}$$

and

$$Y(n, b) = \{x^*\in H^*:\text{ACP}^*(n, x^*, B, A^*, H^*) \text{ is solvable}\}.$$

Define $X=S(\infty, \infty)$ and $Y=Y(\infty, \infty)$.

Suppose X is dense in H and Y is $\sigma(H^*, H)$ dense in H^*. Then A and A^* admit unique solutions for the ACP and ACP* respectively, and $T(t)$ is closable in H with domain containing X, where $T(t)x=y(t, x)$ for $x \in X$ and $y(t, x)$ the corresponding solution. Define

$$X(0, b) = \{x \in H : x \in D(T(t)) \text{ for all } 0 \leqq t \leqq b \text{ and } T(t)x \text{ is continuous in } H\}$$

and

$$Y(0, b) = \{x^* \in H^* : x^* \in D(T(t)^*) \text{ for all } 0 \leqq t \leqq b \text{ and } T(t)^*x^* \text{ is strongly continuous}\}.$$

Let $0 \leqq c \leqq \infty$. The function $x^* \to \sup\{\|A^{*k}T(t)^*\| : 0 \leqq t \leqq c, 0 \leqq k \leqq n\}$ induces a Banach norm on $Y(n, c)$. By the $\sigma(H^*, H)$ denseness of Y in H^*, H can be naturally identified as a subspace of $Y(n, c)^*$. Denote by $H(n, c)$ the closure of H in $Y(n, c)^*$ with the subspace topology. By $H(n, \infty)$, we mean the complete, barreled, bornological, locally convex space which is the completion of H with the topology induced by the intersection of all the subspace topologies from $H(k, c)$ for all $c \geqq 0$ and integers $0 \leqq k < n+1$. We summarize some of the properties of $H(n, b)$ in the following.

THEOREM 2. *Suppose X is dense in H and Y is $\sigma(H^*, H)$ dense in H^*. Then there exists a family of Banach spaces $H(n, b)$ for $0 \leqq b < \infty$ and integers $0 \leqq n < \infty$, and a family of complete, barreled, bornological* l.c.s. *$H(n, \infty)$ for all integers $0 \leqq n \leqq \infty$ such that*

(i) *$H(0, 0) = H$;*

(ii) *A can be extended as a continuous linear transformation from $H(n, c)$ to $H(n+1, a)$ for all $0 \leqq c \leqq a \leqq \infty$ and integers $0 \leqq n \leqq \infty$;*

(iii) *$H(n, b)^* = Y(n, b)$ for all $0 \leqq b \leqq \infty$ and integers $0 \leqq n \leqq \infty$;*

(iv) *for each $0 \leqq t \leqq a-c$ and $0 \leqq n < m < \infty$, $T(t)$ can be extended as a continuous linear transformation from $H(n, c)$ to $H(m, a)$, and $T(t)$ is $m-n-1$ times strongly continuous differentiable;*

(v) *for each $0 \leqq n \leqq \infty$, $T(t)$ can be extended as a continuous linear transformation from E_n to E_n, and for each $x \in H$, $T(t)x$ is n times continuously differentiable in E_n.*

4. As an application of the constructions above, we state the following theorems which can be proved by modifying some of the ideas in Fujiwara **[11]**, Chazarain **[4]** and Beals **[2]**, **[3]**. We say that a closed operator admits an extended semigroup structure if X is dense in H and Y is $\sigma(H^*,H)$ dense in H^*. Let $C(A) = \{$All the continuous operators on H commuting with $A\}$.

THEOREM 3. *Suppose there exists a real number w such that for all λ with* $\operatorname{Re} \lambda > w$, *$R(\lambda, A)$ exists, and* $\|R(\lambda, A)\| \leqq M(1+|\lambda|)^k$ *for some $M>0$ and some positive integer $k \geqq 0$. Then*

(i) *A admits an extended semigroup structure.*

(ii) *For each integer $n \geqq 0$, E_n can be identified as a subset of H_{n+k+2}, and $X(n, \infty)$ contains H_{-n-k-2}.*

(iii) *For each integer n, consider $T(t)$ as a continuous linear transformation from H_n to H_{n+k+2}. Then $T(t)$ is in the closure of $C(A)$ ($\subseteq L(H_n, H_{n+k+2})$), and there exists $M_n>0$ such that $\|T(t)\|_{H_n \to H_{n+k+2}} \leqq M_n e^{rt}$ for all $r>w$ and $0 \leqq t < \infty$.*

THEOREM 4. *Suppose there exist real numbers M, $\alpha>0$, $\beta \geqq 1$ and an integer $k \geqq 0$ such that for all λ with* $\operatorname{Re} \lambda \geqq \alpha \log \beta(1+|\lambda|)$, *$R(\lambda, A)$ exists, and $\|R(\lambda, A)\| \leqq M(1+|\lambda|)^k$.*

(i) *A admits an extended semigroup structure;*

(ii) *for each integer $n \geqq 0$ and $m \geqq 1$, $H(n, t)$ can be identified as a subset of H_{k+n+m}, and $X(n, t)$ contains H_{-k-n-m} for all $0 \leqq t < (n-1)/\alpha$.*

THEOREM 5. *Suppose there exist constants $\alpha, \beta>0$ and $1>\delta>0$, such that $R(\lambda, A)$ exists for all λ with* $\operatorname{Re} \lambda \geqq \alpha|\lambda|^\delta + \beta$, *and for all $\varepsilon>0$ there exists $M, \gamma>0$ such that* $\|R(\lambda, A)\| \leqq M \exp(\varepsilon \operatorname{Re} \lambda + \gamma|\lambda|^\delta)$. *Then A admits an extended semigroup structure.*

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