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ON LOBATCHEWSKY MANIFOLDS

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Let M be a complete, simply connected, n-dimensional Riemannian manifold with sectional curvature $K \leqq 0$. Eberlein in [7] and [9] has given the cone topology and a nice compactification $\bar{M}=M \cup M(\infty)$ of M. The boundary $M(\infty)$ of M is the set of asymptotic classes of geodesics in M. $\bar{M}$ is homeomorphic to the closed unit ball in $\boldsymbol{R}^n$ and $M(\infty)$ is homeomorphic to S^{n-1}. Each isometry ϕ of M extends to a homeomorphism of $\bar{M}$. Elements of the isometry group $I(M)$ can be classified according to their fixed points in $\bar{M}$. ϕ is called elliptic if ϕ has a fixed point in M. ϕ is called parabolic or axial if ϕ has exactly one fixed point or two fixed points in $M(\infty)$ respectively. If any two distinct points in the boundary $M(\infty)$ can be joined by a unique geodesic in M (Axioms I and II), then M is called a Lobatchewsky manifold for convenience. A complete, simply connected Riemannian manifold with sectional curvature $K \leqq c < 0$ is a Lobatchewsky manifold.

In the sequel, we shall consider only Lobatchewsky manifolds M and we shall assume that $I(M)$ acts effectively on M.

The main theorem is a description of complete homogeneous Riemannian manifolds with sectional curvature $K \leqq c < 0$.

THEOREM 1. *Let M be a complete homogeneous Riemannian manifold with sectional curvature $K \leqq c < 0$. Either $I(M)$ has a common fixed point in $M(\infty)$ or M is a noncompact symmetric space of rank one.*

The tool of this paper is the concept of the limit set of a subgroup G of $I(M)$. The limit set $L(G)$ is the intersection with $M(\infty)$ of the closure of any orbit of G in M. The limit set is independent of the choice of the orbit. If A is a closed subset of $M(\infty)$ which contains more than one point and A is invariant under a subgroup G of $I(M)$, then $A \supset L(G)$. The totally geodesic hull $\langle A \rangle$ of a subset A in $M(\infty)$ is the intersection of all totally geodesic submanifolds in M whose boundaries contain A.

Let G be a subgroup of $I(M)$. One obtains classification of $L(G)$ in the following manner: (1) $L(G)$ is empty, (2) $L(G)$ contains one point,

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(3) $L(G)$ contains two points, (4) $L(G)$ is an infinite, perfect and nowhere dense subset of $M(\infty)$, (5) $L(G)=M(\infty)$. Consequently, one can obtain classification of subgroups of $I(M)$ according to their limit sets. A concrete classification of connected Lie subgroups of simple Lie groups of rank one has been accomplished in **[5]** and **[6]**.

Here we present a unified version of the result in **[5]**, **[6]** independent of Cartan's classification.

THEOREM 2. *Let M be a noncompact symmetric space of rank one. Let G be a connected Lie subgroup of $I_0(M)$. Then one of the following holds*:

(1) *G has a common fixed point in M*;

(2) *G has a common fixed point in $M(\infty)$*;

(3) *G modulo a normal subgroup* (*isomorphic to a subgroup of $O(n-1)$*) *is the* 1*-parameter group of axial elements*[2] *along the geodesic joining two fixed points*;

(4) *G modulo a normal subgroup* (*isomorphic to a subgroup of $O(n-m)$, $m=\dim\langle L(G)\rangle$*) *is the connected isometry group $I_0(S)$ of the totally geodesic submanifold $S=\langle L(G)\rangle$ which is a noncompact symmetric space of rank one*;

(5) $G=I_0(M)$.

A consequence of Theorem 2 is the following

THEOREM 3. *Let M be a noncompact symmetric space of rank one and G be a subgroup of $I_0(M)$. If there is no point in $\bar{M}$ and no proper totally geodesic submanifold in M invariant under G, then G is either discrete or dense in $I_0(M)$.*

The above fact is related to Borel's density theorem **[4]** and Selberg's irreducible lattices **[19]**.

We outline the proof of Theorems 1 and 2 by stating two main lemmas.

LEMMA 1. *Let M be a simply connected complete Riemannian manifold with $K\leqq c<0$ such that $I(M)$ acts effectively on M. Suppose that G is a subgroup of $I(M)$ and $\langle L(G)\rangle=M$. If $L(G)$ contains more than two points, then the centralizer $Z(G, I(M))$ of G in $I(M)$ is trivial. If, in addition, G does not have a common fixed point in $M(\infty)$, then G is semisimple.*

LEMMA 2. *Let M be a noncompact symmetric space of rank one and G be a Lie subgroup of $I_0(M)$ such that $M=\langle L(G)\rangle$. Suppose that $L(G)$ contains more than two points and G does not have a common fixed point in $M(\infty)$. Then either G is discrete or $G=I_0(M)$.*

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[2] The factored out normal subgroup of G contains elliptic elements which leave the geodesic pointwise fixed but may rotate other points in M.

Finally we state a theorem on the density of axial fixed points for Lobatchewsky manifolds. This fact is indispensable to geodesic and horospherical G-partition flows on a Lobatchewsky manifold. One can easily obtain a straightforward generalization of **[10]**. Furthermore one gets a corollary which generalizes a theorem **[8]** of Eberlein.

THEOREM 4. *Let M be a Lobatchewsky manifold and G be a subgroup of $I(M)$. If G contains axial elements and G does not have a common fixed point in $M(\infty)$, then the fixed points of axial elements of G are dense in $L(G) \times L(G)$.*

COROLLARY. *Let M be a Lobatchewsky manifold and G be a subgroup of $I(M)$. If G does not have a common fixed point in $M(\infty)$ and $L(G)$ contains more than two points, then G contains a free group with an infinite number of generators.*

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