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AN IMPLICIT FUNCTION THEOREM FOR SMALL DIVISOR PROBLEMS

BY E. ZEHNDER[1]

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1. **Introduction.** Various nonlinear problems, not accessible to standard existence theorems, led to new techniques which allowed the solution of the isometric embedding problem (J. Nash [**1**]) and stability problems of Hamiltonian systems connected with small divisors (A. N. Kolmogorov, V. I. Arnold, J. Moser [**2**]–[**6**]). Subsequently, the underlying ideas were abstracted as implicit function theorems [**7**]–[**10**], which however do not cover most small divisor problems. It is the aim of this paper to formulate and prove a simple implicit function theorem also covering many of these problems. The underlying idea is due to H. Rüssmann [**11**]. Its basic idea is a modification of Newton's method in the framework of linear spaces and not in that of the group of coordinate transformations as it was used in [**2**]–[**6**], [**14**]. The proof of this theorem is elementary; the real difficulty, however, lies in showing that the assumptions can be met in the relevant applications. We mention as a new application the perturbation theory of invariant tori of dimension $m\leqq n$ of globally Hamiltonian diffeomorphisms defined on a $2n$-dimensional symplectic manifold, in which we were able to verify those assumptions. The proof will be published elsewhere. I am indebted to J. Moser for acquainting me with small divisor problems.

2. **Implicit function theorem.** The following set up is prompted by H. Jacobowitz [**9**] and L. Nirenberg [**10**]. We consider three one-parameter families of Banach spaces X_σ, Y_σ, Z_σ in the closed unit interval: for $0\leqq\sigma'\leqq\sigma\leqq 1$,

$$X_0 \supseteq X_{\sigma'} \supseteq X_\sigma \supseteq X_1 \tag{1}$$

(and analogous for Y_σ and Z_σ) and with norms $|\ |_\sigma$ in X_σ, $\|\ \|_\sigma$ in Y_σ and $|\ |_\sigma$ in Z_σ satisfying

$$|f|_{\sigma'} \leqq |f|_\sigma, \qquad \|u\|_{\sigma'} \leqq \|u\|_\sigma, \qquad |z|_{\sigma'} \leqq |z|_\sigma \tag{2}$$

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for $f \in X_\sigma$, $u \in Y_\sigma$, $z \in Z_\sigma$ and $0 \leqq \sigma' \leqq \sigma$. In Y_σ we have a second norm $\| \ \|_\sigma$ such that $|u|_\sigma \leqq \|u\|_\sigma$ and $\|u\|_{\sigma'} \leqq \|u\|_\sigma$ for $u \in Y_\sigma$ and $\sigma' \leqq \sigma$. For fixed $N>0$ and $1 \geqq R>0$ we define the open balls

$$N_\sigma = \{f \in X_\sigma \mid |f|_\sigma < N\} \subset X_\sigma, \qquad R_\sigma = \{u \in Y_\sigma \mid |u|_\sigma < R\} \subset Y_\sigma$$

and

$$\hat{R}_\sigma = \{u \in Y_\sigma \mid \|u\|_\sigma < R\} \subset Y_\sigma.$$

Let $F(\cdot\,,\cdot)$ be a mapping into Z_0 which is defined for every $(f, u) \in X_\sigma \times Y_\sigma$ belonging to $N_\sigma \times R_\sigma$ for some $\sigma>0$, and which is continuous as a mapping from $N_\sigma \times R_\sigma$ into $Z_{\sigma'}$ for every $0 \leqq \sigma' \leqq \sigma$. The aim is to solve for a given f in some N_σ, $F(f, u)=0$, u in some $Y_{\sigma'}$, $\sigma'<\sigma$, assuming that $|F(f, u_0)|_\sigma$ is sufficiently small. We make the following hypothesis in which $\alpha, \beta, \gamma, K_1, K_2, K_3>0$ are fixed.

(i) *Taylor formula.* For every σ in $0<\sigma\leqq 1$ and every $f \in N_\sigma$ the mapping $F(f, \cdot)$ from $R_\sigma \subset Y_\sigma$ into $Z_{\sigma'}$, $\sigma'<\sigma$, has a Fréchet derivative $dF(f, u)$ at every $u \in \hat{R}_\sigma$, and for $u, v \in \hat{R}_\sigma$, $Q(f; u, v)=F(f, u)-F(f, v)-dF(f, u)(u-v)$, satisfies:

$$|Q(f; u, v)|_{\sigma'} \leqq (K_1/(\sigma-\sigma')^\alpha)\, |u-v|_\sigma^2. \tag{3}$$

(ii) *Approximative right inverse of $dF(f, u)$.* For every σ in $0<\sigma\leqq 1$ and every $(f, u) \in N_\sigma \times \hat{R}_\sigma$ there is a linear map $\eta(f, u)(\cdot) \in L(Z_\sigma, Y_{\sigma'})$ for every $\sigma'<\sigma$, such that for all $a \in Z_\sigma$

$$|(dF(f, u) \circ \eta(f, u) - \mathbf{1})(a)|_{\sigma'} \leqq \frac{K_2}{(\sigma-\sigma')^{\alpha+\gamma}}\, |F(f, u)|_\sigma \cdot |a|_\sigma \tag{4}$$

and

$$|\eta(f, u)(a)|_{\sigma'} \leqq (K_3/(\sigma-\sigma')^\gamma)\, |a|_\sigma, \tag{5}$$

$$\|\eta(f, u)(a)\|_{\sigma'} \leqq (K_3/(\sigma-\sigma')^{\gamma+\beta})\, |a|_\sigma. \tag{6}$$

THEOREM 1. *Under the above conditions* (i), (ii) *there exist two constants C_1 and C_2 depending on $\alpha, \beta, \gamma, K_1, K_2$ and K_3 such that if $(f, u_0) \in N_\sigma \times \hat{R}_\sigma$ with $\|u_0\|_\sigma \leqq r<R$ for some σ in $0<\sigma\leqq 1$ satisfies $|F(f, u_0)|_\sigma \leqq C_1 \cdot (R-r) \cdot \sigma^q$, $q \geqq 2\gamma+\alpha+\beta$, then there exists a $u_f \in Y_{\sigma/2}$ with $F(f, u_f)=0$. Further u_f satisfies*

$$|u_f - u_0|_{\sigma/2} \leqq C_2 \cdot |(Ff, u_0)|_\sigma \cdot \sigma^{-\gamma} \tag{7}$$

and

$$\|u_f - u_0\|_{\sigma/2} \leqq (R-r)\sigma^{q-\gamma-\beta}. \tag{8}$$

PROOF. We use Newton's method but replace the inverse of $dF(f, u)$ (which need not exist) with the approximate right inverse $\eta(f, u)$ to define a sequence (u_n), $n \in Z_0^+$ which (as we prove) converges to a solution of $F(f, u)=0$. Let $K=\max\{1, K_i\}$, $1 \leqq i \leqq 3$, $\lambda=4$, $\mu=\lambda/2$, $\kappa=\frac{3}{2}$, $s=q^{-1}$ and $\exp(-\xi)=K^{-3} \cdot 2^{-3(q+1)}$. Set

$$\sigma_n = \sigma/2 \cdot (1 + \exp(-\xi s \kappa^n)), \qquad \tau_{n+1} = \tfrac{1}{2}(\sigma_{n+1} + \sigma_n) \qquad \text{for } n = 0, 1, 2, \cdots .$$

We then have $\sigma_{n+1} < \tau_{n+1} < \sigma_n$ and $\lim \sigma_n = \sigma/2$ as $n \to \infty$. We define inductively the sequence (u_n) by u_0 and

$$(9) \qquad u_{n+1} = u_n + v_n, \quad v_n = -\eta(f, u_n)(F(f, u_n)), \qquad n \geqq 0.$$

To simplify the notation we omit the f's in the following. Using induction we prove the statements S_n:

$$(n1) \qquad u_n \in Y_{\sigma_n}, \quad |F(u_n)|_{\sigma n} \leqq \nu(R-r) \cdot \sigma^q \cdot \exp(-\xi \lambda \kappa^n),$$

$$(n2) \qquad v_n \in Y_{\tau n+1}, \quad |v_n|_{\tau n+1} \leqq \nu(R-r) \cdot \sigma^{q-\gamma} \exp(-\xi \mu \kappa^n),$$

$$(n3) \qquad \|v_n\|_{\tau n+1} \leqq \nu(R-r) \cdot \sigma^{q-\gamma-\beta} \exp(-\xi \mu \kappa^n),$$

$$(n4) \quad u_{n+1} \in \hat{R}_{\tau_{n+1}}, \quad \|u_{n+1} - u_0\|_{\tau_{n+1}} \leqq (R-r)[1 - \exp(-\xi \mu/2\kappa^n)],$$

with $0 < \nu \leqq 1$ to be determined later on. S_0 is valid if $|F(f, u_0)|_\sigma \leqq \nu \cdot C_1 \cdot (R-r) \cdot \sigma^q$ with $C_1 = \exp(-2\lambda\xi)$. Assuming the validity of S_i, $i=1, 2, \cdots, n$, we prove S_{n+1}. Since $u_{n+1}, u_n \in \hat{R}_{\tau_{n+1}} \subset \hat{R}_{s_{n+1}}$, it follows for

$$F(u_{n+1}) = -(dF(u_n) \circ \eta(u_n) - \mathbf{1})(F(u_n)) + Q(u_{n+1}, u_n)$$

using (i) and (ii):

$$\begin{aligned} |F(u_{n+1})|_{\sigma n+1} &\leqq \frac{K_2}{(\sigma_n - \sigma_{n+1})^{\alpha+\gamma}} |F(u_n)|_{\sigma n}^2 + \frac{K_1}{(\tau_{n+1} - \sigma_{n+1})^\alpha} |\eta(u_n)(F(u_n))|_{\tau_{n+1}}^2 \\ &\leqq \left\{ \frac{K_2}{(\sigma_n - \sigma_{n+1})^{\alpha+\gamma}} + \frac{K_1 K_3^2}{(\tau_{n+1} - \sigma_{n+1})^\alpha (\sigma_n - \tau_{n+1})^{2\gamma}} \right\} |F(u_n)|_{\sigma n}^2. \end{aligned}$$

From this estimate $(n+1)1$ follows immediately. Using then (9) and (4), (5), (6) one easily verifies $(n+1)(2-4)$. From $(n1)$ we now conclude that $F(f, u_n) \to 0$ in $Z_{\sigma/2}$ as $n \to \infty$. Since $u_{n+1} - u_n = v_n$, it follows from $(n2)$ that (u_n) is a Cauchy sequence in $Y_{\sigma/2}$. Calling $\lim u_n = u$ we conclude from the continuity of F, that $F(f, u)=0$. From $(n3)$ we get for all n

$$\begin{aligned} \|u_n - u_0\|_{\sigma/2} &\leqq \sum_{n=0}^{\infty} \|v_n\|_{\sigma/2} \\ &\leqq 2\sigma^{q-\gamma-\beta}(R-r)\exp(-\xi\mu[\kappa-1]) < (R-r)\sigma^{q-\gamma-\beta} \end{aligned}$$

and hence (8). Similarly one gets (7); given (f, u_0) such that

$$0 < |F(f, u_0)|_\sigma \leqq C_1 \cdot (R - r)\sigma^q$$

one chooses ν such that $|F(f, u_0)|_\sigma = \nu \cdot C_1 \cdot (R-r)\sigma^q$. □

The new idea consists of introducing an approximate right inverse $\eta(f, u)$, see (4), which is an exact right inverse at the solutions of $F(f, u)=0$! For many small divisor problems such an approximate right inverse can be provided if one is dealing with a conjugation problem.

REMARK (SEE [15]). Various modifications of Theorem 1 are possible. Under additional conditions for F and η the above solution $f \to u_f$ of $F(f, u_f)=0$ is Fréchet differentiable even though the problem may not have a unique solution. Uniqueness of the solutions u_f follows from the existence of an approximative left inverse. A similar theorem holds in the framework of spaces of differentiable functions using smoothing operators.

3. **An application.** For motivation and background consult J. Moser [12] and S. Graff [13]. We consider on the manifold $M = T^n \times \boldsymbol{R}^n \times \boldsymbol{R}^m \times \boldsymbol{R}^m$ the real analytic mapping

$$\varphi_0 : (x, y, \xi, \eta) \to (x + \omega + y, y, \Lambda^+(x)\xi, \Lambda^-(x)\eta),$$

where $\Lambda^\pm(x) \in L^{-1}(\boldsymbol{R}^m)$ with $\sup_{x \in T^n}(\|\Lambda^+(x)^{-1}\|, \|\Lambda^-(x)\|) \leqq a < 1$, and where $\omega = (\omega_1, \cdots, \omega_n)$ satisfies

$$|(\omega, k) - l| \geqq \gamma\, |k|^{-\beta}, \tag{10}$$

for all integers (k, l), $k=(k_1, \cdots, k_n) \neq 0$, $0<\gamma<\frac{1}{2}$, $\beta>n$. The question is, when does the invariant torus $T_0 = T^n \times (\zeta=0)$, $\zeta=(y, \xi, \eta)$ survive under a perturbation by a real analytic mapping f defined in an open neighborhood of T_0. With (φ_λ) we denote the family of mappings

$$\varphi_\lambda : (x, y, \xi, \eta) \to (x + \omega + y, y, (\Lambda^+(x) + \lambda^+(x))\xi, (\Lambda^-(x) + \lambda^-(x))\eta),$$

where λ stands for the pair (λ^+, λ^-). For a given mapping $f=(f_i)$, $1 \leqq i \leqq 4$, on M and $\mu = (\mu_0, \mu_1) \in \boldsymbol{R}^n \times L(\boldsymbol{R}^n)$ we define $f_\mu = (f_1, f_2 + \mu_0 + \mu_1 y, f_3, f_4)$. With

$$\Omega_{\sigma\rho} = \Omega_\sigma \times \Omega_\rho = \{x \mid |\text{Im}\, x| < \sigma\} \times \{|\zeta| < \rho\}$$

we denote complex neighborhoods of the torus T_0.

THEOREM 2. *Given φ_0 as above, then for every $\varepsilon>0$ there exists a $\delta>0$, $\delta(\varepsilon, \Omega_{\sigma\rho}, \varphi_0)$, such that for every real analytic mapping f with $|f-\varphi_0|_{\Omega_{\sigma\rho}}<\delta$ there exist two real analytic mappings φ_λ and $\psi : \Omega_{\sigma/2\rho} \to \Omega_{\sigma\rho}$, $\psi = \text{id} + w$, with $w(x, \zeta) = \alpha(x) + (\beta(x), \zeta)$, and there exist constants $\mu \in (\boldsymbol{R}^n, L(\boldsymbol{R}^n))$ such that*

$$\max\{|\mu|, |\alpha|_{\Omega_{\sigma/2}}, |\beta|_{\Omega_{\sigma/2}}, |\lambda^\pm|_{\Omega_{\sigma/2}}\} < \varepsilon \tag{i}$$

and

(ii) $$T\psi \circ T\varphi_\lambda|_{T_0} = Tf_\mu \circ T\psi|_{T_0},$$

where T_0 denotes the complex torus $\Omega_{\sigma/2} \times \{0\}$ and T the tangent functor.

We look at the mappings $f_\mu \circ (\mathrm{id}+w) - (\mathrm{id}+w) \circ \varphi_\lambda = \phi(f, u)$ on $\Omega_{\sigma\rho}$, where (according to the notation in Theorem 2) $u = (\mu, \lambda, \alpha, \beta)$ is an element of the Banach space Y_σ of vector- and matrix-valued real analytic functions defined on the complex torus Ω_σ. One then proves that the functional F, defined by $F(f, u) = (\phi(f, u)|_{\zeta=0}, d(\phi(f, u))|_{\zeta=0})$, d the Jacobian, satisfies the assumption of Theorem 1. If f and φ_0 are globally Hamiltonian mappings (i.e. $f^* \theta - \theta = ds$, with $\theta = \sum_{i=1}^n y_i \, dx_i + \sum_{j=1}^m \eta_j \, d\xi_j$ and s a function defined on an open neighborhood of T_0) then one can show that $\mu = 0$ in Theorem 2.

COROLLARY. *Let f and φ_0 be real analytic globally Hamiltonian mappings such that $|f - \varphi_0|_{\Omega_{\sigma\rho}}$ is sufficiently small, then there exist two mappings ψ and $\varphi_\lambda : \Omega_{\sigma/2\rho} \to \Omega_{\sigma\rho}$ such that*

$$T\psi \circ T\varphi_\lambda|_{T_0} = Tf \circ T\psi|_{T_0},$$

T and T_0 as in Theorem 2. Moreover the local stable and unstable manifolds of the (under f) invariant torus $\psi(T_0)$ are real analytic and Lagrangian.

Using the methods of J. Moser [6], [9] we get a perturbation theory for hyperbolic tori in the differentiable case. The proofs will appear elsewhere.

REFERENCES

1. J. Nash, *The imbedding problem for Riemannian manifolds*, Ann. of Math. (2) **63** (1956), 20–63. MR **17**, 782.

2. A. N. Kolmogorov, *Théorie générale des systèmes dynamiques et mécanique classique*, Proc. Internat. Congress Math. Amsterdam, vol. I, North-Holland, Amsterdam, 1957, pp. 315–333. MR **20** #4066.

3. V. I. Arnol'd, *Small denominators and problems of stability of motion in classical and celestial mechanics*, Uspehi Mat. Nauk **18** (1963), no. 6 (114), 91–192=Russian Math. Surveys **18** (1963), no. 6, 85–191. MR **30** #943.

4. J. Moser, *On invariant curves of area-preserving mappings of an annulus*, Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II, 1962, 1–20. MR **26** #5255.

5. ———, *A new technique for the construction of solutions of nonlinear differential equations*, Proc. Nat. Acad. Sci. U.S.A. **47** (1961), 1824–1831. MR **24** #A2695.

6. ———, *A rapidly convergent iteration method and non-linear partial differential equations*, Ann. Scuola Norm. Sup. Pisa (3) **20** (1966), I: pp. 265–315, II: pp. 499–535. MR **33** #7667; **34** #6280.

7. J. T. Schwartz, *Nonlinear functional analysis*, Lecture notes, 1964, Gordon and Breach, New York, 1969, pp. 33–44.

8. F. Sergeraert, *Un théorème de fonctions implicites sur certains espaces de Fréchet et quelques applications*, Ann. Sci. École Norm. Sup., 4e série, **5** (1972), 599–660.

9. H. Jacobowitz, *Implicit function theorems and isometric embeddings*, Ann. of Math. (2) **95** (1972), 191–225.

10. L. Nirenberg, *An abstract form of the nonlinear Cauchy-Kowalewski theorem*, J. Differential Geometry **6** (1972), 561–576.

11. H. Rüssmann, *Kleine Nenner.* II: *Bemerkungen zur Newtonschen Methode*, Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. (1972), 1–10.

12. J. Moser, *Stable and random motions in dynamical systems with special emphasis on celestial mechanics*, Princeton Univ. Press, Princeton, N.J., 1973.

13. S. Graff, *On the conservation of hyperbolic invariant tori for Hamiltonian systems*, Dissertation, New York University, New York, June, 1971.

14. S. Sternberg, *Celestial mechanics*, Part II, Benjamin, New York, 1969.

15. E. Zehnder, *The Moser-Nash implicit function theorem for small divisor problems* (*to appear*).

School of Natural Sciences, Institute for Advanced Study, Princeton, New Jersey 08540