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ENERGY DECAYS LOCALLY EVEN IF TOTAL ENERGY GROWS ALGEBRAICALLY WITH TIME

BY CLIFFORD O. BLOOM AND NICHOLAS D. KAZARINOFF

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0. **Introduction.** In this note we announce energy decays locally like $t^{-2+\kappa}$ for solutions of hyperbolic equations, with coefficients that depend upon both position and time, in the exterior of star-shaped domains in $\boldsymbol{R}^3$. Here κ is a positive constant, depending on the coefficients, defined explicitly by (8) below. Our results generalize those of Zachmanoglou [**4**]. He considered a class of equations with time-independent coefficients (see (12) below) and proved under hypotheses roughly analogous to ours that in $\boldsymbol{R}^n$ ($n \geqq 3$) energy decays locally like $t^{-1+\mu}$ ($1 > \mu \geqq 0$). A more important difference between the equations we consider here and those considered by Zachmanoglou in [**4**] is that we treat equations with solutions whose total energy may grow algebraically with t while the total energy of solutions of the equations considered in [**4**] is conserved. In [**1**] we proved that the energy of solutions with bounded total energy decays locally like t^{-2}, but under more stringent hypotheses than those used here.

We now set the scattering problem whose solutions we investigate. Let V be the exterior of a closed, bounded subset B of $\boldsymbol{R}^3$, and let n be the outward unit normal to ∂B. We assume that the origin lies interior to B and that $\partial V \equiv \partial B$ is star-shaped:

$$\min_{x\in\partial V} \frac{n \cdot x}{r} \geqq 0, \tag{1}$$

where $x = (x_1, x_2, x_3)$ and $r^2 = x \cdot x$. Let $Q = (V \cup \partial V) \times [0, \infty)$. We use the notation $\nabla^{(3)} = (\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)$, $\nabla = (\nabla^{(3)}, \partial/\partial t)$. We take as given a symmetric 3×3 matrix E, 1×3 matrices a and b, and functions c and d which satisfy the following hypothesis:

(Hypothesis H_1) (a) *b, c, and E are in $C^1(Q)$; a and d are in $C^2(Q)$,*
(b) *for some $d_0 > 0$, $d(x, t) \geqq d_0$ if $(x, t) \in Q$.*

Let the transpose of a matrix M (or m) be M^T (or m^T). We suppose that E is uniformly elliptic in Q, namely that there exist positive constants c_0 and

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C_0 such that

$$C_0 \geqq \max_{|\xi|=1} \xi E \xi^T \geqq \min_{|\xi|=1} \xi E \xi^T \geqq c_0.$$

Finally we adopt the notation

$$A = \begin{pmatrix} E & a^T \\ a & -d \end{pmatrix}, \qquad D = \begin{pmatrix} E_t & b^T \\ b & -c \end{pmatrix},$$

and

$$\hat{(\cdot)} = \frac{(\cdot)}{\min_{|\xi|=1} \xi E \xi^T}.$$

We consider solutions of the mixed initial boundary value problem

$$\nabla(A(\nabla u)^T) + (b - a_t) \cdot \nabla^{(3)} u + \tfrac{1}{2}(d_t - c)u_t = 0 \qquad (x \in V, t > 0), \tag{2}$$

$$u(x, t) = 0 \qquad (x \in \partial V, t \geqq 0), \tag{3}$$

$$u(x, 0) = f(x), u_t(x, 0) = g(x) \qquad (x \in V), \tag{4}$$

where f and g are functions in $C^1(V \cup \partial V)$ with compact support.

1. **Norms and constants.** We use the following norms:

$$N(\cdot) \equiv \max_{V \cup \partial V} |\cdot| \quad \text{and} \quad N'(\cdot) \equiv \max_{t \geqq 0} N(\cdot).$$

We shall assume that the coefficients c, d and the matrix coefficients a, b, and E are such that the following are positive numbers:

$$\begin{aligned} \alpha_1 &= 2N'(rd_r/d) + 2N'(ra_r/d) + N'(r(d_t - c)/d), \\ \alpha_2 &= 2N'(r\hat{E}_r) + 2N'(r\hat{a}_r) + 4N'(r(\hat{a}_t - \hat{b})) + N'(r(\hat{d}_t - \hat{c})) \\ &\quad + 8N(r^{-2})N'[r^4(\tfrac{1}{2}(d_t - c)_t - \nabla^{(3)} \cdot (a_t - b))]N'(\hat{1}), \\ \alpha_1' &= N'(tb/d) + N'(tc/d), \qquad \alpha_2' = N'(t\hat{E}_t) + N'(t\hat{b}). \end{aligned} \tag{5}$$

In addition to the smoothness conditions on the coefficients already imposed, this hypothesis imposes decay rates on the coefficients in (1) and on their time and space derivatives.

Let Ω be defined by the equation

$$\begin{aligned} \{1 + N'(\hat{d}) + N'(r^{-2})N'(\hat{1})[4N'(r^3 d_r) + 2N'(r^3(d_t - c))] \\ + 4(N'(r^{-4}))^{1/2} N'(\hat{1})N'(r^2 a) + 2N'(\hat{a})\}\Omega = \tfrac{1}{2}. \end{aligned} \tag{6}$$

Again we shall assume that the coefficients of (1) are such that the above norms are well defined.

For any $\varepsilon \in (0, 1)$, each $T > 0$, and some small $\delta > 0$, we define

$$\mathscr{D}(T) = \{x \mid r \leqq \varepsilon\Omega T\} \cap V, \tag{7}$$

$$\kappa = (1 - \varepsilon)^{-1}[\max(\alpha_1, \alpha_2) + (1 + (\varepsilon\Omega)^2)\max(\alpha_1', \alpha_2')] \tag{8}$$

(9) $$s = \max[N'(t\hat{E}_t) + N'(t\hat{b}), N'(tb/d) + N'(tc/d)],$$

and

(10) $$q = -1 + \delta + s.$$

Note that if the differential equation (2) is the wave equation, then $\kappa = 0$.

2. **Local energy decay.** We make the following major hypothesis:

(Hypotheses H_2) *The N'-norm of each of the following is a positive number*:

$$r^{3+q}(E - dI),\quad r^{4+q}E_r,\quad r^{4+q}d_r,\quad r^{4+q}(d_t - c),\quad r^{3+q}a,$$
$$r^{4+q}a_r,\quad r^{4+q}(a_t - b),\quad r^{4+q}[2(d_t - c) + d_t - 2\nabla^{(3)} \cdot a],$$
$$r^{5+q}(d_t - c)_t - 2\nabla^{(3)} \cdot (a_t - b),\quad tE_t, r^{4+q}E_t, tb, \quad \text{and} \quad t^{4+q}c.$$

Let $\mathscr{E}_{\text{loc}}(x, t) = \frac{1}{2}\int_{\mathscr{D}} [du_t^2 + \nabla^{(3)}uE(\nabla^{(3)}u)^T]\,dx$, and let $\mathscr{E}(x, 0)$ be the total initial energy associated with u.

THEOREM 1. *Suppose that ∂V is star-shaped, E is uniformly strongly elliptic, Hypothesis* H_1 *is satisfied and the initial data* (f, g) *in* (4) *are smooth and have compact support in V. Then the unique solution to Problems* (2)–(4) *has compact support. Moreover, suppose $T_0 \equiv T_0(\varepsilon)$ is so large that $\partial V \subset \mathscr{D}(T_0)$, where $\mathscr{D}$ is defined by* (7) *and Hypothesis* H_2 *is satisfied by the coefficients of the differential equation* (2). *Then, for each $\varepsilon \in (0, 1)$, there exist positive constants M, K, and κ, with κ defined by* (8), *such that for $T > T_0$*

(11) $$\mathscr{E}_{\text{loc}}(x, T) \leqq \frac{K\mathscr{E}(x, 0)}{T^2}\left\{1 + \left(\frac{T}{T_0}\right)^{\kappa} e^{M/T_0}\left[1 - \left(\frac{T_0}{T}\right)^{\kappa} + \frac{M}{(1 + \kappa)T_0}\right]\right\}.$$

The theorem holds even if for each positive number p there exists a $t > p$ for which the quadratic form associated with the matrix D defined by (1) fails to be negative semidefinite on $V \cup \partial V$. The negative semidefiniteness of D was a major hypothesis in [**1**].

COROLLARY 1. *If the quadratic form associated with D is negative semidefinite on $(V \cup \partial V) \times [T_0, \infty)$, then the energy decay estimate* (11) *holds with α_1' and α_2', which enter into the definition of κ, both zero and with the exponent q in Hypothesis* H_2 *equal to* $-1 + \delta$.

In the case of the wave equation our decay estimate reduces to $\mathcal{O}(t^{-2})$, which is the same as that obtained by C. S. Morawetz in [**2**]. E. C. Zachmanoglou [**4**] proved energy decays locally like $t^{-1+\mu}$ $(1 > \mu \geqq 0)$ for solutions of hyperbolic equations of the form

(12) $$\nabla^{(3)}(E(x)(\nabla^{(3)}u)^T) - c(x)u - d(x)u_{tt} = 0$$

by generalizing the argument used by Morawetz in [3] in treating the wave equation. To establish a faster rate of energy decay as $t \to \infty$ than the rate Zachmanoglou establishes in [4], we have to impose Hypothesis H_2, which is more stringent by a factor of r than his analogous conditions. But the total energy of solutions of equations of the form (2) satisfying Hypothesis H_2 may grow algebraically with t. In [1] we considered equation (2) with $E = E(x, t)$, $c = 0$, and $d = 1$ under the hypotheses that $E_t \leqq 0$, $c_0 \geqq 1$, and for $t \geqq N$ and $r \geqq \varepsilon t + c$, $|r\nabla^{(3)}E| = \mathcal{O}(t^{-2-\delta})$, $|x(E - I)| = \mathcal{O}(t^{-1-\delta})$, and $|rE_r| = \mathcal{O}(t^{-2-\delta})$ for some positive c and δ.

Our methods of proof in [1] and of Theorem 1 are similar; the estimates we use to prove Theorem 1 are much sharper. Both results are based on the divergence identity

$$\begin{aligned} &\nabla[(\alpha \cdot \nabla u)(Aw) - \alpha^T(\nabla uAw)/2 + u(\gamma A + B)w + C^T u^2/2] \\ &\quad \equiv [\alpha \cdot \nabla u + \gamma u](\nabla(Aw)) + (\nabla \cdot C)u^2/2 + u[C + (\nabla\gamma)A + \nabla \cdot B] \cdot \nabla u \\ &\qquad + \nabla u[\gamma A - (\nabla \cdot \alpha)A/2 - (\alpha \cdot \nabla)A/2 + \nabla\alpha A]w, \end{aligned}$$

where B is an antisymmetric 4×4 matrix with $B^{i*} = 0$ $(i = 1, 2, 3)$, $w = (\nabla u)^T$, $B^{4*} = [-(\beta + t^2 r^{-2})x, 0]d$, $\alpha = (2xt, r^2 + t^2)$, $\gamma = 2t$, $C = (2tx\, dr^{-2} + \Delta d, -d - t^2\, dr^{-2})$, β is a solution of

$$(d\beta)_r + 3(d\beta)r^{-1} = 3\, dr^{-1} + t(d_t - c)r^{-1} + t^2\, d_r r^{-2},$$

$\nabla\alpha = (\alpha^j_{x_i})$, and $\Delta d = 2t(b - a_t) - 2a + [(\beta d)_t + (t^2 d)_t r^{-2}]x - 2tx\, dr^{-2}$. We integrate this identity over the space-time domain bounded by the planes $t = T_0$, $t = T\,(T > T_0)$, and the surface $\partial V \times [T_0, T]$, apply the divergence theorem, and estimate carefully to prove Theorem 1.

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DEPARTMENT OF MATHEMATICS, WAYNE STATE UNIVERSITY, DETROIT, MICHIGAN 48202

DEPARTMENT OF MATHEMATICS, STATE UNIVERSITY OF NEW YORK AT BUFFALO, AMHERST, NEW YORK 14226