BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 79, Number 4, July 1973

THE VITALI-HAHN-SAKS AND NIKODYM THEOREMS FOR ADDITIVE FUNCTIONS. II

BY R. B. DARST

Communicated by Robert Bartle, January 10, 1973

ABSTRACT. In this note appropriate versions of the Vitali-Hahn-Saks and Nikodym theorems are established for s-bounded additive set functions with values in an abelian topological group G.

Although we shall use $+$ and 0 to denote addition and identity in both G and R, the real numbers, no confusion should arise. Thus we denote by $\mathscr{U}$ the set of symmetric neighborhoods of 0 in G, and for $U \in \mathscr{U}$ we set $1U = U$ and $(n+1)U = \{x + y : x \in nU, y \in U\}$, $n \in N$, the set of positive integers.

A subset H of G is said to be bounded if for each $U \in \mathscr{U}$ there exists $n \in N$ such that $H \subset nU$.

We suppose that finite subsets of G are bounded.

Let Ω be a nonempty set and let $\mathscr{S}$ be a sigma algebra of subsets of Ω.

A function μ from $\mathscr{S}$ to G is said to be additive if $\mu(\phi) = 0$ and $\mu(E \cup F) + \mu(E \cap F) = \mu(E) + \mu(F)$, $E, F \in \mathscr{S}$.

An additive function μ is said to be s-bounded (cf. [**1**],[**5**]) if $\lim_n \mu(E_n) = 0$ (i.e., for each $U \in \mathscr{U}$ there is $m \in N$ such that $\mu(E_n) \in U$, $n > m$) for each sequence $\{E_n\}$ of pairwise disjoint elements of $\mathscr{S}$.

Notice that if μ is s-bounded, $U \in \mathscr{U}$, and $\{E_n\}$ is a sequence of pairwise disjoint elements of $\mathscr{S}$, then there exists $n \in N$ such that if M is a finite subset of $N^n = \{k \in N : k \geqq n\}$ then $(\sum_{i \in M} \mu(E_i)) \in U$.

An additive function μ is said to be bounded if $\mu(\mathscr{S}) = \{\mu(E) : E \in \mathscr{S}\}$ is a bounded subset of G.

For the case when μ is sigma-additive, versions of our results can be found in [**4**]; for the case where μ is merely additive and $G = R$, one can refer to [**2**].

Our version of Nikodym's theorem, a striking improvement of the principle of uniform boundedness, follows.

THEOREM 1. *Suppose that T is a set of s-bounded functions such that for each $E \in \mathscr{S}$ the set $T(E) = \{\mu(E) : \mu \in T\}$ is bounded, then $T(\mathscr{S}) = \{\mu(E) : \mu \in T, E \in \mathscr{S}\}$ is bounded.*

AMS (MOS) subject classifications (1970). Primary 28-00, 28A45, 46G99.
Key words and phrases. Vitali-Hahn-Saks theorem, Nikodym theorem, s-bounded, additive set function, abelian group.

PROOF. It suffices to consider the case where T is a countable set $\{\mu_k\}$. Suppose on the contrary that $T(\mathscr{S})$ is not bounded; then let $U \in \mathscr{U}$ such that $T(\mathscr{S}) \not\subset nU$, $n \in N$. Since $T(\Omega)$ is bounded, there is $q_0 \in N$ such that $T(\Omega) \subset q_0 U$. Notice that if $\mu_{k_1}(E) \notin (2q_0 + p_1)U$, then $\mu_{k_1}(\Omega - E) \notin (q_0 + p_1)U$; choose such k_1 for $p_1 = 2$. At least one of the restrictions to E and $(\Omega - E)$ behaves like the original problem. Thus, we set $\Omega_1 = E$, $\mathscr{S}_1 = \{E \cap \Omega_1 : E \in \mathscr{S}\}$, $T_1 = \{\mu|\Omega_1 : \mu \in T\}$, and $F_1 = \Omega - E$, if $T_1(\mathscr{S}_1)$ is unbounded; otherwise, $\Omega_1 = \Omega - E, \cdots$. Iterating this process and relabeling if necessary $(k_j \to j)$ we obtain a sequence $\{F_j\}$ of pairwise disjoint elements of $\mathscr{S}$ such that $\mu_k(F_k) \in \{q_k U - [(\sum_{j<k} q_j) + k + 1]U\}$, $k \in N$. Partitioning $\{F_k\}_{k \geqq 2}$ into a sequence of subsequences and using the fact that μ_1 is s-bounded yields a subsequence $\{F_{n_i}\}_{i \geqq 1}$ such that $\mu_1(\mathscr{S} \cap (\bigcup_{i \geqq 1} F_{n_i})) \subset U$. Repeating this process gives us a subsequence $k_1 = 1, k_2 = n_1, \cdots$ such that $\mu_{k_j}(\mathscr{S} \cap (\bigcup_{i>j} F_{k_i})) \subset U, j \in N$. Set $G = \bigcup_i F_{k_i}$ and notice that the contradiction $\mu_{k_i}(G) \notin k_i U$, $i \in N$, follows. Thus Theorem 1 is established.

When T is a one element set, Theorem 1 specializes as follows.

COROLLARY 1. *An s-bounded function is bounded.*

Theorem 1 also permits us to assert that if $\mu_k(E)$ is Cauchy, $E \in \mathscr{S}$, and G is complete, then the additive function μ defined by $\mu(E) = \lim \mu_k(E)$ is bounded; a corollary of the following Vitali-Hahn-Saks theorem asserts that μ is s-bounded.

A sequence $\{\mu_k\}$ of s-bounded functions is said to be uniformly s-bounded (or uniformly additive—cf. [3]) if for each $U \in \mathscr{U}$ and each sequence $\{E_i\}$ of pairwise disjoint elements of $\mathscr{S}$, there exists $m \in N$ such that $\sum_{i \in M} \mu_k(E_i) \in U$ whenever $k \in N$ and M is a finite subset of N^m.

THEOREM 2. *Suppose that $\{\mu_k\}$ is a sequence of s-bounded functions such that $\{\mu_k(E)\}$ is Cauchy for each $E \in \mathscr{S}$. Then $\{\mu_k\}$ is uniformly s-bounded.*

PROOF. Suppose that $\{\mu_k\}$ is not uniformly s-bounded. Then there exists a sequence $\{E_k\}$ of pairwise disjoint elements of $\mathscr{S}$, $U \in \mathscr{U}$, a sequence $\{M_k\}$ of pairwise disjoint finite subsets of N, and an increasing sequence $\{n_k\}$ of elements of N such that

(i) $\mu_{n_k}(F_k) \notin 5U$, where $F_k = \bigcup_{i \in M_k} E_i$,

(ii) $b_k = \max\{i \in M_k\} < \min\{i \in M_{k+1}\} = a_{k+1}$,

(iii) if M is a finite subset of $N_{b_{k-1}} = \{n \in N; n \leqq b_{k-1}\}$, then $(\mu_{n_k} - \mu_j)(\bigcup_{i \in M} E_i) \in U$, $j > n_k$, and

(iv) if M is a finite subset of $N^{a(k+1)}$, then $\mu_{n_k}(\bigcup_{i \in M} E_i) \in U$.

Set $\nu_k = \mu_{n_k} - \mu_{n(k+1)}$; then $\nu_k(E) \to 0$, $E \in \mathscr{S}$, and $\{F_k\}$ is a sequence of pairwise disjoint elements of $\mathscr{S}$ with $\nu_k(F_k) \notin 4U$, $k \in N$. Partition $\{F_i\}_{i \geqq 2}$ into a sequence $\{\{F_{i_j}\}_{j \geqq 1}\}_{i \geqq 1}$ of subsequences. Since ν_1 is s-bounded there is a least integer i_1 such that $i \geqq i_1$ implies that $\{\nu_1(E): E \in \mathscr{S}, E \subset \bigcup_{j=1}^{\infty} F_{ij}\} \subset U$ (i.e., $\{\nu_1|(\bigcup_{j\geqq 1} F_{ij})\} \subset U$). Then since ν_{i_1} is s-bounded we can repeat this process and choose a subsequence of $\{F_{(i_1)_j}\}$ on which ν_{i_1} stays in U. This process may be iterated, after which a diagonalization and relabeling yields (cf. (iii)) $\nu_k(\bigcup_{i\in M\subset N_{k-1}} F_i) \in U$, $\nu_k(F_k) \notin 4U$, and $\nu_k(F) \in U$ if $F \in \mathscr{S}$ and $F \subset (\bigcup_{j>k} F_j)$. Thus, the contradiction $\nu_k(\bigcup_{i\geqq 1} F_i) \notin 2U$, $k \in N$, obtains, and Theorem 2 is established.

An immediate corollary follows.

COROLLARY 2. *If $\{\mu_k\}$ is a sequence of s-bounded functions and $\mu(E) = \lim \mu_k(E)$, $E \in \mathscr{S}$, then μ is s-bounded.*

In conclusion we remark that several generalizations are possible (cf. the theorems in [**4**]).

REFERENCES

1. R. B. Darst, *The Lebesgue decomposition*, Duke Math. J. **30** (1963), 553–556. MR **28** #176.

2. ———, *On a theorem of Nikodym with applications to weak convergence and von Neumann algebras*, Pacific J. Math. **23** (1967), 473–477. MR **38** #6360.

3. ———, *The Vitali-Hahn-Saks and Nikodym theorems for additive set functions*, Bull. Amer. Math. Soc. **76** (1970), 1297–1298. MR **41** #8625.

4. D. Landers and L. Rogge, *The Vitali-Hahn-Saks and uniform boundedness theorem in topological groups*, Manuscripta Math. **4** (1971), 351–359. MR **44** #402.

5. C. E. Rickart, *Decomposition of additive set functions*, Duke Math. J. **10** (1943), 653–665. MR **5**, 232.

DEPARTMENT OF MATHEMATICS, COLORADO STATE UNIVERSITY, FORT COLLINS, COLORADO 80521