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A NOTE ON WITT RINGS

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This note contains some applications of the theory of Mackey functors (cf. [3], [4] and [5]) to the study of Witt rings. A detailed version may be found in [3, Appendices A and B].

So let R be a commutative ring with $1 \in R$ and $W(R)$ its Witt ring as defined in [7]. Any ring homomorphism $\rho: R \to R'$ defines a ring homomorphism $\rho_*: W(R) \to W(R')$. Moreover if R' is separable over R and finitely generated projective as an R-module (let ρ be called admissible in this case), the trace map $R' \to R$ defines a $W(R)$-linear map backwards: $\rho^*: W(R') \to W(R)$ (cf. [1] and [12], [13]). These observations lead easily to

PROPOSITION 1. *Let $\mathfrak{C}$ be the category with objects the commutative rings R (with $1 \in R$) and with morphisms $[R', R]_{\mathfrak{C}} = \{\rho: R \to R' | \rho \text{ admissible}\}$ (i.e. $\mathfrak{C}$ is dual to the category of commutative rings with admissible maps). Then the Witt ring construction defines a Mackey functor $W: \mathfrak{C} \to \mathscr{A}\ell$, the category of abelian groups, together with a commutative, associative and unitary inner composition, given by the multiplication in the Witt ring.*

COROLLARY 1. *Let $\rho: R \to R'$ be admissible and $n \cdot 1_{W(R)} \in \rho^*(W(R'))$ for some $n \in N$. Then all the "Amitsur cohomology groups" $H^i(R'/R, W)$ (i.e. the cohomology groups of the semisimplicial complex $0 \to W(R) \to W(R') \rightrightarrows W(R' \otimes_R R') \Rrightarrow W(R' \otimes_R R' \otimes_R R') \cdots$) are n-torsion groups, especially for $n = 1$ they are all trivial.*

PROOF. Apply the results of [5] to this special situation (they were found precisely to be applied right here!).

Examples of admissible maps $\rho: R \to R'$ with $1_{W(R)} \in \rho^*(W(R'))$ have been given by Scharlau (cf. [12] and [3, Appendix A, Lemmas 2.3, 2.4, 2.5]).

As a rather special case we get this way:

COROLLARY 2 (CF. [11] AND [8]). *Let L/K be a finite Galois extension (of fields) of odd degree and with Galois group G. Then the natural action of G on $W(L)$ has trivial (co)homology:*

$$H^0(G, W(L)) \cong H_0(G, W(L)) \cong W(K),$$

$$H^i(G, W(L)) = H_i(G, W(L)) = \hat{H}^j(G, W(L)) = 0 \qquad (i \geqslant 1, j \in Z).$$

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Let us concentrate further on finite separable field extensions:

For a field K the category of finite, commutative separable K-algebras is (by Galois!) dual to the category $G\hat{}$ of finite G-sets, where G is the (profinite) Galois group of K, thus W defines a Mackey functor on $G\hat{}$ with a commutative, associative and unitary inner composition, i.e. a Green functor in the sense of [**3**, §8]. The theory of Burnside rings and Mackey functors can easily be extended from finite to profinite groups and, applied to W, then leads to

PROPOSITION 2. *There exists a natural ring homomorphism of the Burnside ring $\Omega(G)$ onto $W(K)$ (defined by mapping any transitive G-set G/U (U an open subgroup of G) in $\Omega(G)$ onto the element in $W(K)$, which is represented by the bilinear form (L,t) with L the fixed field of U, considered as an K-vector space, and $t: L \times L \to K: (a, b) \mapsto \mathrm{trace}_{L/K}(a \cdot b)$ the canonical bilinear map, associated with the separable extension L/K), which in case* char $K \neq 2$ *is surjective, already restricted to $\Omega(\bar{G}) \subseteq \Omega(G)$ with $\bar{G}$ the Galois group of $K(\sqrt{a} \mid a \in K)$ over K.*

This at first explains the analogy between the prime ideal structure of Burnside rings (cf. [**2**] and [**3**, §5]) and Witt rings (cf. [**9**]). Furthermore Proposition 2, combined with the results of [**9**] and [**3**, §5], allows us to obtain easily the famous theorems of A. Pfister, concerning the ring theoretic structure of Witt rings, e.g. that all torsion in Witt rings is 2-torsion.

PROPOSITION 3. *Let K be a field and $\rho_i: K \to L_i$ $(i = 1, \ldots, t)$ be finite separable field extensions of K, all contained in the finite Galois extension E/K.*

Then:

$$(1)\quad (E:K)_2 \cdot 1_{W(K)} \subseteq \bigcap_{i=1}^{t} Ke\left(W(K) \overset{\rho_{i*}}{\to} W(L_i)\right) + \sum_{i=1}^{t} \mathrm{Im}\left(W(L_i) \overset{\rho_i^*}{\to} W(K)\right),$$

$$(2)\qquad (E:K)_2 \cdot \left(\bigcap_{i=1}^{t} Ke(\rho_{i*}) \cap \sum_{i=1}^{t} \mathrm{Im}(\rho_i^*)\right) = 0$$

(with $(E:K)_2$ the maximal power of 2, dividing the degree $(E:K)$),

(3) $2^n \cdot 1_{W(K)} \in \sum_1^n \mathrm{Im}(\rho_i^*)$ *for some power 2^n of $2 \Leftrightarrow$ any ordering of K can be extended to at least one of L_i $(i = 1, \ldots, t)$.*

COROLLARY 3. *If L/K is a finite Galois extension with Galois group G, then all the groups $H^i(G, W(L))$, $H_i(G, W(L))$ and $\hat{H}^j(G, W(L))$ $(i \geqq 1, j \in \mathbf{Z})$ are $(L:K)_2$-torsion groups.*

Using J. A. Green's transfer theorem (cf. [**6**]) one gets furthermore:

COROLLARY 4. *Let E/K be a finite Galois extension, L/K a maximal subextension of odd degree and F/K the minimal subextension in L/K such that L/F is normal (i.e. the fixed field of all K-automorphisms of L). Let $L_1, \ldots, L_t$ be a family of subextensions of E/K with $L \subseteq L_i$, which contains the compositions $L \cdot L^\tau$ for any $\tau \in G = \mathrm{Gal}(E/K)$ with $L^\tau \neq L$.*

Then the imbeddings $\rho_i : F \to L_i$, $\sigma_i : K \to L_i$, $\sigma : K \to F$ induce an isomorphism

$$W(K) \Big/ \sum_1^t \mathrm{Im}(\sigma_i^*) \to W(F) \Big/ \sum_1^t \mathrm{Im}(\rho_i^*) : a + \sum_1^t \mathrm{Im}(\sigma_i^*) \mapsto \sigma_*(a) + \sum_1^t \mathrm{Im}(\rho_i^*),$$

whose inverse is given by

$$W(F) \Big/ \sum_1^t \mathrm{Im}(\rho_i^*) \to W(K) \Big/ \sum_1^t \mathrm{Im}(\sigma_i^*) : b + \sum \cdots \mapsto \sigma^*(b) + \sum \cdots .$$

Finally let us remark on a rather curious byproduct of these results: Let G be the full Galois group of a formally real field and $H \leqq G$ an open subgroup of odd index. Then there exists an open subgroup $F \leqq H$ of index 2, such that for any closed subgroup $U \leqq G$ of order 2 the number $|G/F^U| = |\{gF | UgF = gF\}|$ of U-invariant cosets of F in G equals $|G/H^U| + 1$.

REFERENCES

1. F. R. De Mayer, *The trace map and separable algebras*, Osaka J. Math. **3** (1966), 7–11. MR **37** #4122.
2. A. Dress, *A characterization of solvable groups*, Math. Z. **110** (1969), 213–217. MR **40** #1491.
3. ———, *Notes on the theory of representations of finite groups*. I: *The Burnside ring of a finite group and some* AGN-*applications*, Multicopied lecture notes, Bielefeld, 1971.
4. ———, *A note on G-functors*, Bielefeld, 1972 (preprint).
5. ———, *Tate-Amitsur-cohomology of Mackey-functors*, Bielefeld, 1972 (preprint).
6. T. A. Green, *Axiomatic representation theory for finite groups*, J. Pure Appl. Algebra **1** (1971), 41–77.
7. M. Knebusch, *Grothendieck- und Wittringe von nichtausgearteten symmetrischen Bilinearformen*, S.-B. Heidelberger Akad. Wiss. Math.-Natur. K. **1969/70**, 93–157. MR **42** #6001.
8. M. Knebusch und W. Scharlau, *Über das Verhalten der Wittgruppe bei Körpererweiterungen*, Math. Inst. der Universität Münster und Saarbrücken, 1970 (preprint).
9. J. Leicht und F. Lorenz, *Die Primideale des Wittschen Ringes*, Invent. Math. **10** (1970), 82–88. MR **42** #1851.
10. A. Pfister, *Quadratische Formen in beliebigen Körpern*, Invent. Math. **1** (1966), 116–132. MR **34** #169.
11. A. Rosenberg and R. Ware, *The zerodimensional Galois cohomology of Wittrings*, Invent. Math. **11** (1970), 65–72.
12. W. Scharlau, *Zur Pfisterschen Theorie der quadratischen Formen*, Invent. Math. **6** (1969), 327–328. MR **39** #2793.
13. ———, *Induction theorems and the structure of the Wittgroup*, Invent. Math. **11** (1970), 37–44.

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