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SELFADJOINT SUBSPACE EXTENSIONS OF NONDENSELY DEFINED SYMMETRIC OPERATORS[1]

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1. **Subspaces in $\mathfrak{H}^2$.** Let $\mathfrak{H}$ be a Hilbert space over the complex field C, and let $\mathfrak{H}^2 = \mathfrak{H} \oplus \mathfrak{H}$ be the Hilbert space of all pairs $\{f, g\}$, where $f, g \in \mathfrak{H}$, with the inner product $(\{f, g\}, \{h, k\}) = (f, h) + (g, k)$. A *subspace* T in $\mathfrak{H}^2$ is a closed linear manifold in $\mathfrak{H}^2$; its *domain* $\mathfrak{D}(T)$ is the set of all $f \in \mathfrak{H}$ such that $\{f, g\} \in T$ for some $g \in \mathfrak{H}$, and its *range* $\mathfrak{R}(T)$ is the set of all $g \in \mathfrak{H}$ such that $\{f, g\} \in T$ for some $f \in \mathfrak{H}$. For $f \in \mathfrak{D}(T)$ we put $T(f) = \{g \in \mathfrak{H} | \{f, g\} \in T\}$. A subspace T in $\mathfrak{H}^2$ is the graph of a linear function if $T(0) = \{0\}$; in this case we say T is an *operator* in $\mathfrak{H}$, and then we denote $T(f)$ by Tf.

The *adjoint* T^* of a subspace T in $\mathfrak{H}^2$ is defined by

$$T^* = \{\{h, k\} \in \mathfrak{H}^2 | (g, h) = (f, k) \text{ for all } \{f, g\} \in T\}.$$

If J is the unitary operator in $\mathfrak{H}^2$ given by $J\{f, g\} = \{g, -f\}$, then $T^* = \mathfrak{H}^2 \ominus JT$, the orthogonal complement of JT in $\mathfrak{H}^2$. This shows that T^* is also a subspace in $\mathfrak{H}^2$.

If T is a subspace in $\mathfrak{H}^2$, let $T_\infty = \{\{f, g\} \in T | f = 0\}$. Then $T_s = T \ominus T_\infty$ is a closed operator in $\mathfrak{H}$, and we have the orthogonal decomposition $T = T_s \oplus T_\infty$, with $\mathfrak{D}(T_s)$ dense in $\mathfrak{H} \ominus T^*(0)$, $\mathfrak{R}(T_s) \subset \mathfrak{H} \ominus T(0)$.

A *symmetric subspace* S in $\mathfrak{H}^2$ is one satisfying $S \subset S^*$, and a *selfadjoint subspace* H is a symmetric one such that $H = H^*$. If $H = H_s \oplus H_\infty$ is a selfadjoint subspace in $\mathfrak{H}^2$ we have the result (due to Arens, [**1**, Theorem 5.3]) that H_s, considered as an operator in $\mathfrak{H} \ominus H(0)$, is a densely defined selfadjoint operator there. This permits a spectral analysis of a selfadjoint subspace H, once its operator part H_s and its purely multi-valued part H_∞ have been identified.

If S, S_1 are symmetric subspaces in $\mathfrak{H}^2$ such that $S \subset S_1$, then S_1 is said to be a *symmetric extension* of S. In [**3**] (see also [**2**]) we described all symmetric and selfadjoint extensions of a symmetric subspace S in $\mathfrak{H}^2$. In this note we characterize precisely, in terms of "generalized boundary conditions", those selfadjoint subspace extensions of a nondensely defined symmetric operator S in $\mathfrak{H}$. Applications to ordinary differential operators will be indicated in a subsequent note. Detailed

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proofs will appear elsewhere.

We require from [**3**, Theorems 12 and 15] two characterizations of the selfadjoint extensions H of a symmetric subspace S in $\mathfrak{H}^2$. All such satisfy $S \subset H \subset S^*$; let $M = S^* \ominus S$.

THEOREM A. *A subspace H is a selfadjoint extension of S in $\mathfrak{H}^2$ if and only if $H = S \oplus M_1$, where M_1 is a subspace of M satisfying $JM_1 = M \ominus M_1$.*

Alternatively, such H may be described in terms of the subspaces $M^{\pm} = \{\{h, k\} \in S^* | k = \pm ih\}$. We have $M = M^+ \oplus M^-$ and the following result.

THEOREM B. *A subspace H is a selfadjoint extension of S in $\mathfrak{H}^2$ if and only if there exists an isometry V of M^+ onto M^- such that $H = S \oplus (I - V)M^+$, where I is the identity operator. Thus S has a selfadjoint extension in $\mathfrak{H}^2$ if and only if* $\dim M^+ = \dim M^-$.

2. **Selfadjoint extensions of nondensely defined symmetric operators.** Let S_0 be a symmetric densely defined operator in $\mathfrak{H}$, and let $\mathfrak{H}_0$ be a subspace of $\mathfrak{H}$. Throughout this section we *assume* that

$$\dim \mathfrak{H}_0 = p < \infty, \qquad \dim M_0 < \infty, \qquad M_0 = S_0^* \ominus S_0. \tag{2.1}$$

We define S to be the operator in $\mathfrak{H}$ given by

$$\mathfrak{D}(S) = \mathfrak{D}(S_0) \cap (\mathfrak{H} \ominus \mathfrak{H}_0), \qquad S \subset S_0. \tag{2.2}$$

This operator is not densely defined, and so its adjoint will be a subspace which is not an operator.

THEOREM 1. *Let S be defined by* (2.2), *where* (2.1) *is assumed. Then S is a symmetric operator with $\mathfrak{D}(S)$ dense in $\mathfrak{H} \ominus \mathfrak{H}_0$, and*

$$S^* = \{\{h, S_0^*h + \varphi\} | h \in \mathfrak{D}(S_0^*), \varphi \in \mathfrak{H}_0\}, \tag{2.3}$$

$$\dim M^{\pm} = \dim(M_0)^{\pm} + \dim \mathfrak{H}_0. \tag{2.4}$$

Thus $S^*(0) = \mathfrak{H}_0$ and S^* is the algebraic sum of S_0^* and $(S^*)_\infty$. From (2.4) and Theorem B it follows that S has selfadjoint extensions in $\mathfrak{H}^2$ if and only if $\dim(M_0)^+ = \dim(M_0)^-$, that is, if and only if S_0 has selfadjoint extensions in $\mathfrak{H}$. We now *assume* $\dim(M_0)^+ = \dim(M_0)^- = \omega$, and indicate how one can characterize any selfadjoint extension H of S in $\mathfrak{H}^2$ by means of "generalized boundary conditions". Theorem A implies that any such $H = S \oplus M_1$ can be thought of as $H = S^* \ominus JM_1$, where $\dim M_1 = p + \omega$. Thus

$$H = \{\{h, k\} \in S^* | (k, \alpha) - (h, \beta) = 0 \text{ for all } \{\alpha, \beta\} \in M_1\},$$

and (2.3) implies that H is the set of all $\{h, S_0^*h + \varphi\} \in S^*$ satisfying

(2.5) $$\langle h\alpha\rangle - (h, \varphi') + (\varphi, \alpha) = 0$$

for all $\{\alpha, S_0^*\alpha + \varphi'\} \in M_1$. Here we have introduced the abbreviation $\langle h\alpha\rangle = (S_0^*h, \alpha) - (h, S_0^*\alpha)$, $h, \alpha \in \mathfrak{D}(S_0^*)$. By thinking of H as in (2.5) we obtain the following precise characterization.

THEOREM 2. *Let H be a selfadjoint subspace extension of S in $\mathfrak{H}^2$, with* $\dim H(0) = s$. *Let an orthonormal basis for $H(0)$ be $\varphi_1, \ldots, \varphi_s$, and suppose $\varphi_1, \ldots, \varphi_s, \varphi_{s+1}, \ldots, \varphi_p$ is an orthonormal basis for $S^*(0) = \mathfrak{H}_0$. Then H is the set of all $\{h, S_0^*h + \varphi\} \in S^*$ such that*

(i) $(h, \varphi_j) = 0, j = 1, \ldots, s$,

(ii) $\langle h\delta_j\rangle - (h, \zeta_j) = 0, j = p + 1, \ldots, p + \omega$,

(iii) $\varphi = c_1\varphi_1 + \cdots + c_s\varphi_s + \sum_{k=s+1}^{p} [(h, \psi_k) - \langle h\gamma_k\rangle]\varphi_k$, $c_j \in C$ *arbitrary*,

where

(a) $\gamma_{s+1}, \ldots, \gamma_p \in \mathfrak{D}(S_0^*)$,

(b) $\delta_{p+1}, \ldots, \delta_{p+\omega} \in \mathfrak{D}(S_0^*)$ *are linearly independent* mod $\mathfrak{D}(S_0)$, *and* $\langle \delta_j\delta_k\rangle = 0, j, k = p + 1, \ldots, p + \omega$,

(c) $\zeta_j = -\sum_{k=s+1}^{p} \langle \delta_j\gamma_k\rangle\varphi_k, j = p + 1, \ldots, p + \omega$,

(d) $\psi_j = \sum_{k=s+1}^{p} [E_{kj} - \frac{1}{2}\langle \gamma_j\gamma_k\rangle]\varphi_k, j = s + 1, \ldots, p$, $E_{jk} \in C$, $E = (E_{jk}) = E^*$.

Conversely, let $\varphi_1, \ldots, \varphi_p$ be an orthonormal basis for $\mathfrak{H}_0$, suppose γ_j, δ_j exist satisfying (a), (b), *and ζ_j, ψ_j are defined by* (c), (d). *Then H defined via* (i)–(iii) *is a selfadjoint extension of S with* $\dim H(0) = s$.

The operator part H_s of H is given by

$$H_sh = Q_0S_0^*h + \sum_{k=s+1}^{p} [(h, \psi_k) - \langle h\gamma_k\rangle]\varphi_k,$$

where Q_0 is the orthogonal projection of $\mathfrak{H}$ onto $\mathfrak{H} \ominus H(0)$.

With appropriate interpretations, Theorem 2 remains valid in the three cases: $s = 0$, $s = p$, and $\omega = 0$. If $s = 0$ then H is an operator extension of S, and those operator extensions H satisfying $S_0 \subset H \subset S_0^*$ are obtained by taking $\gamma_j = 0$, $E_{kj} = 0$, which results in $\zeta_j = 0$, $\psi_j = 0$. Then

$$\mathfrak{D}(H) = \{h \in \mathfrak{D}(S_0^*) | \langle h\delta_j\rangle = 0, j = p + 1, \ldots, p + \omega\},$$

$$\langle \delta_j\delta_k\rangle = 0, \qquad j, k = p + 1, \ldots, p + \omega,$$

which is the known characterization of such H. If $\omega = 0$ and $s = p$, $H(0) = \mathfrak{H}_0$ and $H_sh = Q_0S_0h$. Thus, given any selfadjoint operator S_0 in $\mathfrak{H}$, with $\mathfrak{D}(S_0)$ dense in $\mathfrak{H}$, and subspace $\mathfrak{H}_0 \subset \mathfrak{H}$, $\dim \mathfrak{H}_0 < \infty$, the operator H_s on $\mathfrak{H} \ominus \mathfrak{H}_0$ defined by $H_sh = Q_0S_0h$ is a densely defined selfadjoint operator. This is a result due to W. Stenger [**4**, Lemma 1].

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