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MAPPINGS INTO HYPERBOLIC SPACES

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In this note we state some results on extensions of holomorphic mappings into hyperbolic spaces. A theorem involves extending holomorphic mappings to a domain of holomorphy. An extension problem of holomorphic mappings into a taut complex space was considered by Fujimoto [**1**].

Another result is that the space of all meromorphic mappings from a complex space X into a hyperbolically imbedded space in Y is relatively compact in the space of all meromorphic mappings from X into Y.

A relatively compact complex space M is said to be hyperbolically imbedded in a complex space Y if for all sequences $\{p_n\}$ and $\{q_n\}$ in M such that $p_n \to p \in \overline{M}$ and $q_n \to q \in \overline{M}$ and such that $d_M(p_n, q_n) \to 0$, we have $p = q$. Here d_M denotes the pseudo-distance defined by Kobayashi [**5**]. A relatively compact complex space M in Y is strictly Levi pseudoconvex if for every point $p \in \partial M$ there are a neighborhood U_p of p and a biholomorphic map Φ_p of U_p onto a subvariety of a domain D_p in some C^n and a function φ defined in U_p such that $\varphi \circ \Phi_p^{-1}$ is the restriction to $\Phi_p(U_p)$ of a strictly pluri-subharmonic function $\tilde{\varphi}_p$ defined in D_p and $\Phi_p(U_p \cap M) = \{x \in \Phi(U_p) : \tilde{\varphi}_p(x) < 0\}$.

THEOREM 1. *Let X be a complex manifold and A be an analytic subset of X of codimension at least* 1. *Let M be a strictly Levi pseudoconvex hyperbolic space in Y. Then a holomorphic mapping f of $X - A$ into M can be extended holomorphically to a mapping $\tilde{f}$ of X into M.*

This theorem can be proved using a theorem by Kwack [**6**] and the fact that there exist a neighborhood W of ∂M and a pluri-subharmonic function ψ defined on W such that $W \cap M = \{x \in M : \psi(x) < 0\}$.

THEOREM 2. *Let M be one of the following*: (i) *M is a hyperbolic and strictly Levi pseudoconvex subspace of a complex space Y, and* (ii) *M is a complex manifold having a complete Hermitian metric ds_M^2 all of whose holomorphic sectional curvatures are nonpositive. Let N be an* (*unramified*) *Riemann domain over a Stein manifold and f be a holomorphic mapping of N into M. Then the existence domain of the mapping f from N into M is a Stein manifold.*

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COROLLARY. *Let f, N, M, and Y be as above and $H(N)$ be the envelope of holomorphy. Then $f:N \to M$ can be extended to a holomorphic mapping $\tilde{f}$ from $H(N)$ into M.*

Theorem 2 was proved when M is a taut complex space by Fujimoto [**1**]. The proof of Theorem 2 uses the following lemma and arguments used by Fujimoto.

LEMMA 1. *Let p be a point in $\mathbf{C}^n$, $n \geqq 2$. Consider hyperspheres B with the center p and S whose boundary contains p. Let D be the intersection of the interior of B and the exterior of S. Then every holomorphic mapping f of D into M has a holomorphic extension to a neighborhood of p where M is as in Theorem 2.*

This lemma is essentially proved by Griffiths [**2**] when M is a complex manifold having a complete Hermitian metric all of whose holomorphic sectional curvatures are nonpositive.

We can also prove

THEOREM 3. *Let X be a complex space Y with the following property; every holomorphic mapping of the punctured disk in $\mathbf{C}$ into X extends holomorphically to a mapping of the whole disk into X. Then if $f:M \to X$ is a holomorphic mapping of an (unramified) Riemann domain M over a Stein manifold, the existence domain of f is a Stein manifold and $f:M \to X$ can be extended to a holomorphic mapping $\tilde{f}:H(M) \to X$ where $H(M)$ denotes the domain of holomorphy of M.*

Next we state a convergence theorem for a sequence of meromorphic mappings. (See [**7**] for definition.)

THEOREM 4. *Let X be a complex space and $\{f_n\}$ be a sequence of meromorphic mappings of X into a hyperbolically imbedded space in Y. Then there is a subsequence which converges uniformly on compact subsets to a meromorphic mapping $f:X \to Y$.*

If X is a complex manifold without singularities, then it is known that any meromorphic mapping f of X into a hyperbolic space is holomorphic and in this case Theorem 4 follows from theorems by Kiernan [**3**].

The proof of Theorem 4 uses the following theorem of Bishop [**8**].

THEOREM (BISHOP). *The limit of a sequence of purely k-dimensional analytic varieties whose $2k$-volumes are uniformly bounded is again a purely k-dimensional variety.*

We also use the following theorem by Kiernan [**4**].

THEOREM (KIERNAN). *M is hyperbolically imbedded in Y if and only if* *for each Hermitian metric h on Y, there exists a constant* $c > 0$ *such that* $f^*(ch) \leqq ds_D^2$ *for every holomorphic mapping of a unit disk D in* **C** *into M.*

Theorem 4 may be proved using a resolution of a complex space as used by Kiernan.

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