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QUASINILPOTENT OPERATORS AS NORM LIMITS OF NILPOTENT OPERATORS

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In [**1**, Problem 7], P. R. Halmos raised this problem: Is every quasinilpotent operator the norm limit of nilpotent operators? Since a limit of nilpotent operators may fail to be quasinilpotent, he noted that a better formulation of the problem would be: What is the closure of the set of nilpotent operators? In attempts at answers, details of which are in [**4**], [**5**], we studied T, an algebra of operators on $C[0, 1]$ [**2**, pp. 701–702]. T is radical and hence every element in it is quasinilpotent. Using $\| \|_1$, the sup norm, under which T is a Banach algebra, we show that the closure of the set of nilpotent elements in T is a maximal ideal. Hence any operator in T not in this ideal is a quasinilpotent operator which, under the sup norm, is not the norm limit of nilpotent operators. However, we show that under an incomplete multiplicative norm for T, $\| \|_2$, every element in T is the norm limit of nilpotent operators. Under both $\| \|_1$ and $\| \|_2$ there are sequences of nonnilpotent quasinilpotent operators in T which converge to nonzero nilpotent operators.

There is a discontinuous isomorphism of the algebra T under $\| \|_2$ onto T under $\| \|_1$. Moreover, if there exists a complete multiplicative norm for T under which every operator is the norm limit of nilpotent ones, then there exists a discontinuous isomorphism of T under this norm onto T under the sup norm. Thus the problem is related to the study of the continuity of automorphisms of nonsemisimple Banach algebras which was urged by I. Kaplansky, who suggested that nontrivial radical algebras might be investigated [**3**, p. 3].

R. G. Douglas (cited in [**1**, p. 916]) has shown that every compact quasinilpotent operator is the limit of nilpotent ones. The question of whether every quasinilpotent operator is either nilpotent or compact is raised and answered in the negative in [**1**, pp. 916, 918]. Our results, combined with Douglas' result, provide an alternative proof that there exist quasinilpotent operators which are neither nilpotent nor compact.

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