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# STOCHASTIC INTEGRALS AND PARABOLIC EQUATIONS IN ABSTRACT WIENER SPACE

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Kuo [**2**] has developed a theory of stochastic integrals and Piech [**3**] has established the existence of fundamental solutions of a class of parabolic equations, both working within the context of abstract Wiener space. In this note we establish the relationship between the work of Kuo and Piech, and as a consequence of this relationship we obtain a uniqueness theorem for fundamental solutions. We also provide a new proof of the non-negativity and semigroup properties of fundamental solutions.

Let $H$ be a real separable Hilbert space, with inner product $( \ , \ )$ and norm $|\cdot|$; let $\|\cdot\|$ be a fixed measurable norm on $H$; let $B$ be the completion of $H$ with respect to $\|\cdot\|$; and let $i$ denote the natural injection of $H$ into $B$. The triple $(H, B, i)$ is an abstract Wiener space in the sense of Gross [**1**]. We may regard $B^* \subset H^* \approx H \subset B$ in the natural fashion. A bounded linear operator from $B$ to $B^*$ may thus be viewed as an operator on $B$ or, by restriction to $H$, as an operator on $H$. The restriction to $H$ of a member $T$ of $L(B, B^*)$ is of trace class in $L(H)$ $(\equiv L(H, H))$ and

$$\|T_{|H}\|_{\mathrm{Tr}} \leqq \text{constant} \cdot \|T\|_{L(B,B^*)}.$$

Where no confusion of interpretation is possible, we will use $T$ for $T_{|H}$. In order to work with stochastic integrals on $(H, B, i)$ we formulate the following hypothesis:

(h) There exists an increasing sequence $\{P_n\}$ of finite dimensional projections on $B$ such that $P_n[B] \subset B^*$, $\{P_n\}$ converges strongly to the identity on $B$, and $\{P_{n|H}\}$ converges strongly to the identity on $H$.

For $t > 0$, let $p_t$ denote the Wiener measure on the Borel field of $B$ which is determined by Gauss cylinder set measure on $H$ of variance parameter $t$. Let $\Omega$ be the space of continuous functions $\omega$ from $[0, \infty)$ into $B$ and vanishing at zero, and let $\mathcal{M}$ be the $\sigma$-field of $\Omega$ generated by the functions $\omega \to \omega(t)$. Then there is a unique probability measure $\mathcal{P}$ on $\mathcal{M}$ for which the condition $0 = t_0 < t_1 < \cdots < t_n$ implies that $\omega(t_{j+1}) - \omega(t_j)$, $0 \leqq j \leqq n - 1$, are independent and $\omega(t_{j+1}) - \omega(t_j)$ has distribution

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measure $p_{t_{j+1}-t_j}$ in $B$. The process $W_t$ defined by $W_t(\omega) \equiv \omega(t)$ is called a Wiener process on $B$. The following theorem is a special case of Theorem 5.1 of [**2**].

THEOREM 1. *Assume that $C$ satisfies the following conditions:*

(1-a) $C: B \to L(B)$;

(1-b) $C(x) - I$ *has range in* $B^*$ *for all* $x$ *in* $B$;

(1-c) $\|C(x) - C(y)\|_{H-S} \leqq \text{constant} \cdot \|x - y\|_B$ *for all* $x$ *and* $y$ *in* $B$; *where* $\|\cdot\|_{H-S}$ *is the Hilbert-Schmidt norm in* $L(H)$;

(1-d) $\|C(x) - I\|_{H-S}^2 \leqq \text{constant} \cdot (1 + \|x\|)^2$ *for all* $x$ *in* $B$. *Then the stochastic integral equation*

$$X_t(\omega) = X_0(\omega) + \int_0^t C(X_s(\omega))\, dW_s(\omega)$$

*possesses a unique continuous solution which is nonanticipating with respect to the family* $\{\mathscr{M}_t\}$ *where* $\mathscr{M}_t$ *is the* $\sigma$*-field generated by* $\{W_s: 0 \leqq s \leqq t\}$. *This solution is a homogeneous strong Markov process.*

Assume that $f$ is a function with domain in $B$ and range in some Banach space $W$. The Fréchet derivative of $f$ at $x$ will be denoted by $f'(x)$ and is a member of the space $L(B, W)$. The $H$-derivative of $f$ at $x$ will be denoted by $Df(x)$ and is the value at zero of the Fréchet derivative of the function $g: H \to W$ defined by $g(h) \equiv f(x + h)$.

We consider a differential operator of the form

$$L_{x,t}\, u(x, t) \equiv \text{trace}\, [A(x)D^2u(x, t)] - \partial/\partial t\, u(x, t)$$

where $A: B \to L(H)$, $u: B \times (0, \infty) \to R$ and $D$ denotes $H$-differentiability, for $t$ fixed. We say that $L_{x,t}u$ exists if the relevant derivatives exist and if $A(x)D^2u(x, t)$ is of trace class in $L(H)$. We may now state the results of [**3**].

THEOREM 2. *Assume that* $A(x)$ *is of the form* $I - B(x)$, *where*

(2-a) $B(x)$ *is a symmetric member of* $L(H)$ *and there exists an* $\varepsilon > 0$ *such that* $B(x) \leqq (1 - \varepsilon)I$ *for all* $x$ *in* $B$;

(2-b) *there exists a symmetric Hilbert-Schmidt operator* $E$ *on* $H$ *such that* $B(x)$ *is of the form* $EB_0(x)E$, *where* $B_0(x) \in L(H)$ *and* $\|B_0(x)\|_{L(H)} \leqq 1$ *for all* $x$ *in* $B$;

(2-c) $B_0''(x)$ *exists and is a bounded uniformly Lip-1 function from* $B$ *to* $L(B \to L(B \to L(H)))$;

(2-d) $|B_0'(x)|_{L(B \to L(H))}$ *is uniformly bounded*;

(2-e) *for any orthonormal basis* $\{e_i\}$ *of* $H$, $\sum_i |B_0'(x)e_i|_{L(H)}^2 <$ *constant, independently of* $x$ *in* $B$.

*Then there exists a family of finite real-valued signed Borel measures* $\{q_t(x, dy): 0 < t < \infty, x \in B\}$ *on* $B$ *such that if*

$$q_t f(x) \equiv \int_B f(y) q_t(x, dy),$$

*then for each bounded real-valued uniformly Lip-1 function $f$ on $B$ we have $L_{x,t} q_t f(x) = 0$ for all $x$ in $B$ and $t > 0$. Moreover $\|q_t f - f\|_\infty \to 0$ as $t \downarrow 0$.*

Assume henceforth that hypothesis (h) holds and that $A(x)$ satisfies (2-a)–(2-e). We require in addition that $B(x)$ is the restriction to $H$ of an operator which we also denote by $B(x)$ and which satisfies

(2-f) $B(\cdot): B \to L(B, B^*)$.

We may now regard $A(\cdot): B \to L(B)$. Then for each $x$ in $B$ $A(x)_{|H}$ is positive definite and symmetric by (2-a). Therefore $[A(x)_{|H}]^{1/2}$ exists as a member of $L(H)$. Moreover $I + [A(x)_{|H}]^{1/2}$ is invertible in $L(H)$. We define

$$A(x)^{1/2} \equiv I - \{I + [A(x)_{|H}]^{1/2}\}^{-1} B(x).$$

It is easy to see that $A(x)^{1/2}$ satisfies (1-a), (1-c) and (1-d). (1-b) will follow once we establish that $[I + [A(x)_{|H}]^{1/2}](B^*) = B^*$. Writing $[A(x)_{|H}]^{1/2} = I - B(x)\{I + [A(x)_{|H}]^{1/2}\}^{-1}$ we see that $[A(x)_{|H}]^{1/2}$ maps $B^*$ to $B^*$ and $H \backslash B^*$ to $H \backslash B^*$. Since $I + [A(x)_{|H}]^{1/2}$ is invertible in $L(H)$ it follows that $[I + [A(x)_{|H}]^{1/2}](B^*) = B^*$. Since $C(x) \equiv A(x)^{1/2}$ satisfies (1-a)–(1-d) the stochastic integral equation

$$X_t(\omega) = X_0(\omega) + \int_0^t [A(X_s(\omega))]^{1/2}\, dW_s(\omega) \tag{1}$$

has a unique solution $X_t$. We define

$$r_t(x, dy) \equiv \mathscr{P}\{X_t \in dy : X_0 = x\}. \tag{2}$$

THEOREM 3. *The fundamental solution $\{q_t(x, dy)\}$ of Theorem 1 coincides with the family $\{r_{2t}(x, dy)\}$ of transition probabilities associated with the solution of (1) and defined by (2). That is, $q_t(x, dy) = r_{2t}(x, dy)$ for all $t > 0$ and $x$ in $B$.*

PROOF. Two families of finite Borel measures on $B$ are identical if they act identically on all bounded real-valued uniformly Lip-1 functions $f$. That is, for any such $f$, we must show that

$$\int_B f(y) q_{t/2}(x, dy) = \int_B f(y) r_t(x, dy). \tag{3}$$

We will write the left side of (3) as $q_{t/2} f(x)$. Fix $\tau > 0$. Define $F: [0, \tau) \times B \to R$ by $F(t, x) = q_{(\tau - t)/2} f(x)$. Then by Theorem 1 the function

$$g(t, x) \equiv \partial/\partial t\, F(t, x) + \tfrac{1}{2} \operatorname{trace} A(x) D^2 F(t, x)$$

is identically zero on $[0, \tau) \times B$. It will be proved in a forthcoming paper

[5] that, for each bounded real-valued Lip-1 function $f$ on $B$, the maps $(t, x) \to D(q_t f)(x)$ from $(0, \infty) \times B$ to $H$ with $|\cdot|$ and $(t, x) \to D^2(q_t f)(x)$ from $(0, \infty) \times B$ to the space of trace class operators on $H$ with trace class norm are continuous. This enables us to apply Ito's formula [2, Theorem 4.1] to $F(t, x)$, obtaining

$$F(t, X_t(\omega)) = F(0, x) + \int_0^t g(s, X_s(\omega))\, ds$$

$$+ \int_0^t \langle [A(X_s(\omega))]^{1/2} DF(s, X_s(\omega)), dW_s(\omega) \rangle \tag{4}$$

$$= q_{\tau/2} f(x) + \int_0^t \langle [A(X_s(\omega))]^{1/2} DF(s, X_s(\omega)), dW_s(\omega) \rangle$$

for $0 \leqq t < \tau$. $\langle\ ,\ \rangle$ denotes the $B^*$–$B$ pairing. By [2, (4) of Theorem 3.2] the expectation ($\mathscr{E}$) of the second term on the right side of (4) is zero. Thus

$$\mathscr{E}[F(t, X_t(\omega))] = q_{\tau/2} f(x).$$

Letting $t \uparrow \tau$, we obtain

$$\int_B f(y) r_\tau(x, dy) = \mathscr{E}[f(X_\tau(\omega))] = q_{\tau/2} f(x).$$

This establishes (3) and proves the theorem.

REMARK. Since the measures $\{q_t(x, dy)\}$ form the transition probabilities of a Markov process, it is an immediate consequence that $q_s q_t f(x) = q_{s+t} f(x)$ (the "semigroup property") and that $q_t(x, dy)$ is a probability measure. These properties cannot be easily deduced from the work in [3]. They have been established in [4] in the presence of additional hypotheses of a technical nature on $A(x)$ (and in the absence of (2-f) and hypothesis (h)).

We note that, for the proof of Theorem 3, we have used only the properties of $q_t f$ mentioned in the statement of Theorem 2 together with smoothness properties of $Dq_t f$ and $D^2 q_t f$. We have thus proved the following uniqueness result for the fundamental solution of $L_{x,t} u = 0$.

THEOREM 4. *Assume that* $L_{x,t}$ *satisfies* (2-a)–(2-f) *and that* $B$ *satisfies hypothesis* (h). *Then the family* $\{q_t(x, dy): t > 0, x \in B\}$ *whose existence is asserted by Theorem 2 is unique among families* $\{\mu_t(x, dy): t > 0, x \in B\}$ *of bounded real-valued signed Borel measures on B which satisfy the following requirements:*

*For each bounded real-valued uniformly* Lip-1 *function* $f$ *on* $B$, *setting* $\mu_t f(x) \equiv \int_B f(y) \mu_t(x, dy)$,

(4-a) $\mu_t f(x)$ *satisfies* $L_{x,t} \mu_t f(x) = 0$, (4-b) $\|\mu_t f - f\|_\infty \to 0$ *as* $t \downarrow 0$,

(4-c) $(t, x) \to D(\mu_t f)(x)$ *and* $(t, x) \to D^2(\mu_t f)(x)$ *are continuous from* $B \times (0, \infty)$ *to* $H$ *and to the space of trace class operators on* $H$ *respectively.*

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