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PERIODIC AND HOMOGENEOUS STATES ON A VON NEUMANN ALGEBRA. II[1]

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This paper is a natural continuation of the previous paper [**9**]. In [**9**], we proved a structure theorem for a von Neumann algebra with a fixed periodic and homogeneous state. In this paper, we will show that the structure theorem in [**9**] determines intrinsically the algebraic type of a factor with a periodic and *inner* homogeneous state (see Definition 1). We keep the terminologies and the notations in [**9**].

DEFINITION 1. A normal state φ on a von Neumann algebra $\mathscr{M}$ is said to be *inner homogeneous* if $G(\varphi) \cap \operatorname{Int}(\mathscr{M})$ acts ergodically on $\mathscr{M}$, that is, if the group of all inner automorphisms of $\mathscr{M}$ leaving φ invariant has no fixed points other than the scalar multiples of the identity.

For each $a \in \mathscr{M}$, we write

$$\operatorname{Ad}(a)x = axa^*, \qquad x \in \mathscr{M}.$$

Since $\operatorname{Ad}(u) \in G(\varphi)$ for a unitary $u \in \mathscr{M}$ if and only if u falls in $\mathscr{M}_\varphi$, the centralizer of φ, the inner homogeneity of φ is equivalent to the fact that $\mathscr{M}_\varphi' \cap \mathscr{M} = \{\lambda 1\}$. Hence $\mathscr{M}_\varphi$ is a II_1-factor and $\mathscr{M}$ itself is also a factor.

We consider two periodic and inner homogeneous faithful normal states φ and ψ on $\mathscr{M}$. We denote by $\{\mathscr{M}_n^\varphi : n = 0, \pm 1, \ldots\}$ and $\{\mathscr{M}_n^\psi : n = 0, \pm 1, \ldots\}$ the decompositions of $\mathscr{M}$ in [**9**, Theorem 11] corresponding to φ and ψ respectively. By [**9**, Theorem 13], φ and ψ have the same period, say $T > 0$. Let $\kappa = e^{-2\pi/T}$, $0 < \kappa < 1$.

Following Connes' idea, we consider the tensor product $\mathscr{P} = \mathscr{M} \otimes \mathscr{L}(\mathfrak{H}_2)$ of $\mathscr{M}$ and the 2×2-matrix algebra $\mathscr{L}(\mathfrak{H}_2)$. Let $\{e_{i,j} : i, j = 1, 2\}$ be a system of matrix units in $\mathscr{L}(\mathfrak{H}_2)$. Every $x \in \mathscr{P}$ is of the form

$$x = x_{11} \otimes e_{11} + x_{12} \otimes e_{12} + x_{21} \otimes e_{21} + x_{22} \otimes e_{22},$$

where $x_{ij} \in \mathscr{M}$. We define a faithful state χ on $\mathscr{P}$ by

$$\chi(x) = \tfrac{1}{2}(\varphi(x_{11}) + \psi(x_{22})).$$

Connes showed in [**3**] that there exists a strongly continuous one-

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parameter family $\{u_t\}$ of unitaries in $\mathcal{M}$ such that $\sigma_t^\chi(1 \otimes e_{12}) = u_t \otimes e_{12}$ and $\sigma_t^\psi(x) = u_t^* \sigma_t^\varphi(x) u_t$, $x \in \mathcal{M}$.

LEMMA 2. $u_{s+t} = \sigma_s^\varphi(u_t)u_s$.[2]

PROOF. The above cocycle equality follows from the simple calculation.

$$\begin{aligned} u_{s+t} \otimes e_{12} &= \sigma_{s+t}^\chi(1 \otimes e_{12}) = \sigma_s^\chi \cdot \sigma_t^\chi(1 \otimes e_{12}) \\ &= \sigma_s^\chi(u_t \otimes e_{12}) = \sigma_s^\chi((u_t \otimes e_{11})(1 \otimes e_{12})) \\ &= (\sigma_s^\varphi(u_t) \otimes e_{11})(u_s \otimes e_{12}) = \sigma_s^\varphi(u_t)u_s \otimes e_{12}. \end{aligned}$$

Since $\sigma_T^\varphi = \sigma_T^\psi = \mathrm{id}$, u_T must be a scalar multiple of 1, so that one can find $1 \leqq \alpha < e^{2\pi/T}$ with $u_T = \alpha^{iT}1$. Let $v_t = \alpha^{-it}u_t$. We have then the following properties:

$$v_t^* \sigma_t^\varphi(x) v_t = \sigma_t^\psi(x), \qquad x \in \mathcal{M}; \tag{1}$$

$$v_{s+t} = \sigma_s^\varphi(v_t)v_s = \sigma_t^\varphi(v_s)v_t; \tag{2}$$

$$v_T = 1. \tag{3}$$

We define a one-parameter family $\{\rho_t\}$ of isometries of $\mathcal{M}$ onto $\mathcal{M}$ by

$$\rho_t(x) = \sigma_t^\varphi(x)v_t, \qquad x \in \mathcal{M}. \tag{4}$$

From (2) and (3) we obtain the following:

$$\rho_{s+t} = \rho_s \cdot \rho_t; \tag{5}$$

$$\rho_T = \mathrm{id}. \tag{6}$$

For each integer n, let $\mathcal{V}_n = \{x \in \mathcal{M} : \rho_t(x) = \kappa^{int}x\}$. Since ρ_t has period T, $\mathcal{V}_n \neq \{0\}$ for some n.

LEMMA 3. *An $x \in \mathcal{M}$ falls in $\mathcal{V}_n$ if and only if*

$$\sigma_t^\chi(x \otimes e_{12}) = \alpha^{it}\kappa^{int}(x \otimes e_{12}).$$

PROOF. We compute as follows:

$$\begin{aligned} \sigma_t^\chi(x \otimes e_{12}) &= \sigma_t^\chi((x \otimes e_{11})(1 \otimes e_{12})) \\ &= (\sigma_t^\varphi(x) \otimes e_{11})(u_t \otimes e_{12}) = \sigma_t^\varphi(x)u_t \otimes e_{12} \\ &= \alpha^{it}\sigma_t(x)v_t \otimes e_{12} = \alpha^{it}\rho_t(x) \otimes e_{12}. \end{aligned}$$

Thus, the assertion follows.

Therefore, we conclude that $x \in \mathcal{V}_n$ if and only if $\alpha\kappa^n\chi(y(x \otimes e_{12}))$

[2] ADDED IN PROOF. This cocycle equation is mentioned in the final version of [3], which was missed from an earlier version and not available at the time when this article was finished.

$= \chi((x \otimes e_{12})y)$ for every $y \in \mathscr{P}$.

LEMMA 4. *If* $x \in \mathscr{V}_n$ *and* $y \in \mathscr{M}$, *then we have*

$$\alpha\kappa^n\psi(yx) = \varphi(xy). \tag{7}$$

PROOF. For each $x \in \mathscr{V}_n$ and $y \in \mathscr{M}$, we have

$$\begin{aligned}\alpha\kappa^n\psi(yx) &= 2\alpha\kappa^n\chi((y \otimes e_{21})(x \otimes e_{12}))\\ &= 2\chi((x \otimes e_{12})(y \otimes e_{21})) \quad \text{by Lemma 3}\\ &= \varphi(xy).\end{aligned}$$

LEMMA 5. $\mathscr{M}_m^\varphi \mathscr{V}_l \mathscr{M}_n^\psi \subset \mathscr{V}_{m+l+n}$.

PROOF. For each $a \in \mathscr{M}_m^\phi$, $b \in \mathscr{M}_n^\psi$ and $x \in \mathscr{V}_l$, we have

$$\begin{aligned}\rho_t(axb) &= \sigma_t^\varphi(axb)v_t = \kappa^{imt}a\sigma_t^\varphi(x)\sigma_t^\varphi(b)v_t\\ &= \kappa^{imt}a\sigma_t^\varphi(x)v_tv_t^*\sigma_t^\varphi(b)v_t\\ &= \kappa^{imt}a\kappa^{ilt}x\kappa^{int}b = \kappa^{i(m+l+n)t}axb.\end{aligned}$$

Since $\mathscr{M}_m^\varphi$, $m \geqq 1$, contains isometries and $\mathscr{M}_n^\psi$, $n \leqq -1$, contains co-isometries by [**9**], we conclude that $\mathscr{V}_n \neq \{0\}$ for every integer n. Since $\mathscr{M}_0^\varphi$ and $\mathscr{M}_0^\psi$ are both factors, we conclude that, for every pair of nonzero projections $p \in \mathscr{M}_0^\varphi$ and $q \in \mathscr{M}_0^\psi$,

$$p\mathscr{V}_n q \neq \{0\}. \tag{8}$$

For an $x \in \mathscr{V}_n$, let $x = uh = ku$ be the right and left polar decomposition of x. We have then

$$k(\kappa^{int}u) = \kappa^{int}x = \rho_t(x) = \sigma_t^\varphi(x)v_t = \sigma_t^\varphi(k)\sigma_t^\varphi(u)v_t.$$

By the unicity of the polar decomposition, we get $k = \sigma_t^\varphi(k)$ and $\kappa^{int}u = \rho_t(u)$; hence $u \in \mathscr{V}_n$. Similarly, we get $h = \sigma_t^\psi(h)$.

Thus, we obtain the following:

LEMMA 6. *For every* $x \in \mathscr{V}_n$, *we have*

$$xx^* \in \mathscr{M}_0^\varphi \quad \text{and} \quad x^*x \in \mathscr{M}_0^\psi.$$

If u is a partial isometry in $\mathscr{V}_n$, then we have, by (7),

$$\varphi(uxu^*) = \alpha\kappa^n\psi(xu^*u), \qquad x \in \mathscr{M}.$$

Taking fact (8) into account and making use of the usual exhaustion arguments, we conclude that there exists an isometry or a co-isometry u in $\mathscr{V}_n$ according as $\alpha\kappa^n \leqq 1$ or $\alpha\kappa^n > 1$. Considering various n, we conclude the following:

THEOREM 7. *For periodic inner homogeneous faithful states φ and ψ on a factor $\mathcal{M}$, there exist isometries u and v in $\mathcal{M}$ such that*

$$\psi(x) = \varphi(uxu^*)/\varphi(uu^*),$$
$$\varphi(x) = \psi(vxv^*)/\psi(vv^*), \qquad x \in \mathcal{M}; \tag{9}$$

$$uu^* \in \mathcal{M}_0^\varphi \quad \textit{and} \quad vv^* \in \mathcal{M}_0^\psi. \tag{10}$$

Fixing a periodic inner homogeneous state φ on a factor $\mathcal{M}$, for each projection $p \in \mathcal{M}_0^\varphi$ we define a state φ_p on $p\mathcal{M}p$ by

$$\varphi_p(x) = \varphi(x)/\varphi(p), \qquad x \in p\mathcal{M}p.$$

By Theorem 7, any other periodic inner homogeneous state ψ on $\mathcal{M}$ is unitarily equivalent to φ_p for some $p \in \mathcal{M}_0^\varphi$; more precisely, there exists an isometry $u \in \mathcal{M}$ with $uu^* = p \in \mathcal{M}_0^\varphi$ such that $\psi(x) = \varphi_p(uxu^*)$ for every $x \in \mathcal{M}$. Thus, the set $\{\varphi_p : p \in \mathcal{M}_0^\varphi\}$ exhausts all possible periodic inner homogeneous states up to unitary equivalence.

THEOREM 8. *Let p and q be two projections in $\mathcal{M}_0^\varphi$, and let u and v be isometries in $\mathcal{M}$ with $uu^* = p$ and $vv^* = q$. Let $\psi(x) = \varphi_p(uxu^*)$ and $\omega(x) = \varphi_q(vxv^*)$, $x \in \mathcal{M}$. Then ψ and ω are unitarily equivalent if and only if $\varphi(p) = \kappa^n\varphi(q)$ for some integer n.*

THEOREM 9. *For the state ψ defined in the previous theorem, there exists $\sigma \in \mathrm{Aut}(\mathcal{M})$ with $\psi = \varphi \circ \sigma$ if and only if there exists an isomorphism ρ of $\mathcal{M}_0^\varphi$ onto $p\mathcal{M}_0^\varphi p$ and a partial isometry w in $\mathcal{M}_0^\varphi$ such that $\rho \cdot \theta(x) = w\theta \cdot \rho(x)w^*$, $x \in \mathcal{M}_0^\varphi$, where θ denotes the isomorphism described in* [**9**, *Theorem* 11].

By the following theorem, one can distinguish the algebraic type of $\mathcal{M}$ in terms of $\{\mathcal{M}_0^\varphi, \theta\}$.

THEOREM 10. *Let $\mathcal{M}$ and $\mathcal{N}$ be factors equipped with periodic inner homogeneous states φ and ψ respectively. Let $\{\mathcal{M}_0, \theta\}$ and $\{\mathcal{N}_0, \rho\}$ be the relevant couples of* II_1*-factors and isomorphisms described in* [**9**, *Theorem* 11] *respectively. Let $e_{-1} = \theta(1)$ and $f_{-1} = \rho(1)$. Necessary and sufficient conditions for $\mathcal{M}$ and $\mathcal{N}$ to be isomorphic are that* (i) *$\varphi(e_{-1}) = \psi(f_{-1})$;* (ii) *there exists an isomorphism σ of $\mathcal{N}_0$ onto $p\mathcal{M}_0 p$ for some projection p with $p \geqq e_{-1}$ and a partial isometry w in $\mathcal{M}_0$ such that $w\theta \cdot \sigma(x)w^* = \sigma \cdot \rho(x)$, $x \in \mathcal{N}_0$.*

Making use of the new results of Connes in [**5**], we can prove the following:

THEOREM 11. *Let $\mathcal{M}$ be a factor equipped with a periodic homogeneous state. The existence of a periodic inner homogeneous state on $\mathcal{M}$ is equiva-*

lent to the fact that $S(\mathcal{M}) \neq \{0, 1\}$, where $S(\mathcal{M})$ means the invariant of $\mathcal{M}$ defined by Connes in [**1**].

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