BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 79, Number 2, March 1973

SPECTRAL MAPPING THEOREMS ON A TENSOR PRODUCT

BY ROBIN HARTE

Communicated by Robert G. Bartle, August 17, 1972

1. **Introduction.** By computing the joint spectrum [5], [6] for certain systems of elements in a tensor product [3], [11] of Banach algebras, and applying the spectral mapping theorem in several variables [5], [6], [7], we find that we can determine the spectrum of certain linear operators, notably the tensor product $S \otimes T$ discussed by Brown and Pearcy [1], [12]. We can also see that the spectrum of an "operator matrix" [4], [10] is what it ought to be, and recover the results of Lumer and Rosenblum [10] about the multiplication operators $L_S R_T$ and $L_S + R_T$. Full proofs, and more detail, will appear elsewhere [8].

2. **Left and right spectra.** Suppose that A is a complex Banach algebra, with identity 1. Then the *joint spectrum* of a system of elements $a \in A^n$ is the union of the *left spectrum* and the *right spectrum* [5, Definition 1.1]:

$$\sigma_A^{\text{joint}}(a) = \sigma_A^{\text{left}}(a) \cup \sigma_A^{\text{right}}(a) \tag{2.1}$$

where

$$\sigma_A^{\text{left}}(a) = \left\{ s \in C^n : 1 \notin \sum_{j=1}^{n} A(a_j - s_j) \right\} \tag{2.2}$$

and

$$\sigma_A^{\text{right}}(a) = \left\{ s \in C^n : 1 \notin \sum_{j=1}^{n} (a_j - s_j)A \right\}. \tag{2.3}$$

The *spectral mapping theorem* [5, Theorem 3.2] is the equality

$$\sigma_A^{\text{joint}} f(a) = f \sigma_A^{\text{joint}}(a), \tag{2.4}$$

valid for a commuting system of elements $a \in A^n$ and a system $f = (f_1, f_2, \ldots, f_m)$ of polynomials in n complex variables. Equality (2.4) is also valid for left and right spectra separately; it extends [7, Theorem 4.2] to certain noncommuting systems of elements, where of course the idea of a "polynomial" has to be extended. Here we take a "polynomial in n variables" to be an element of the free complex algebra-with-identity Poly_n on n generators z_j; for an arbitrary system of elements $a \in A^n$, the mapping $f \to f(a) : \text{Poly}_n \to A$ is a homomorphism which preserves

AMS (MOS) subject classifications (1970). Primary 47D99, 46H99; Secondary 47A10, 46M05.

Key words and phrases. Joint spectrum, spectral mapping theorem, tensor product, uniform crossnorm, operator matrix, multiplication operator.

identity and sends each z_j into the corresponding a_j, and then a system $f = (f_1, f_2, \ldots, f_m) \in \mathrm{Poly}_n^m$ defines a mapping $f: A^n \to A^m$.

It will be convenient, for what follows, if we summarize the spectral mapping theorems for a composite system of elements $(a, b) \in A^{n+m}$ associated with two systems $a \in A^n$ and $b \in A^m$. It is also convenient here to work explicitly with the left spectrum (2.2): The arguments for the right spectrum are obviously exactly similar, and can be obtained formally by "reversing products" in the algebra A; then we obtain usually the corresponding statement for the joint spectrum by taking unions.

As a convenient abbreviation, write [7, Definition 1.1]

$$\sigma^{\mathrm{left}}_{a=s}(b) = \{t \in \sigma^{\mathrm{left}}(b) : (s, t) \in \sigma^{\mathrm{left}}(a, b)\}, \tag{2.5}$$

for arbitrary systems of elements $a \in A^n$, $b \in A^m$ and scalars $s \in C^n$. Also

$$\sigma^{\mathrm{left}}_{a=a}(b) = \bigcup \{\sigma^{\mathrm{left}}_{a=s}(b) : s \in \sigma^{\mathrm{left}}(a)\}. \tag{2.6}$$

LEMMA 1 [7, *Theorem* 2.3]. *If* $a \in A^n$, $b \in A^m$, $s \in C^n$ *and* $f \in \mathrm{Poly}^p_{n+m}$, *and if each* a_j *commutes with each* b_k, *then there is equality*

$$\sigma^{\mathrm{left}}_{a=s} f(a, b) = \sigma^{\mathrm{left}}_{a=s} f(s, b). \tag{2.7}$$

THEOREM 1 [5, *Theorems* 3.2, 4.2, 4.3]. *If* $a \in A^n$, $b \in A^m$ *and* $f \in \mathrm{Poly}^p_{n+m}$, *then with no restriction there is inclusion*

$$f\sigma^{\mathrm{left}}(a, b) \subseteq \sigma^{\mathrm{left}} f(a, b). \tag{2.8}$$

If $a \in A^n$ *is commutative and commutes with* $b \in A^m$ *then there is equality*

$$\sigma^{\mathrm{left}} f(a, b) = \sigma^{\mathrm{left}}_{a=a} f(a, b). \tag{2.9}$$

If the whole system $(a, b) \in A^{n+m}$ *is commutative then there is equality*

$$\sigma^{\mathrm{left}} f(a, b) = f\sigma^{\mathrm{left}}(a, b). \tag{2.10}$$

These results are valid [7, Theorems 4.2, 4.3] if we replace each commutivity condition by the corresponding "quasi-commutivity" requirement [7, Definition 3.1].

3. **Tensor products.** If A and B are complex Banach algebras then we shall denote by $A \otimes B$ the completion of the algebraic "tensor product" $A \otimes_C B$ with respect to some *uniform crossnorm* [3], [11] which is compatible with the multiplication $(a \otimes b)(a' \otimes b') = (aa') \otimes (bb')$. Thus elements of the form $\sum_{r=1}^R a_r \otimes b_r$ form a dense subspace, elements of the form $a_1 \otimes b_1$ have norm $\|a_1\| \, \|b_1\|$, and for every pair of bounded linear functionals $\varphi \in A^*$ and $\psi \in B^*$, the linear functional

$$\varphi \otimes \psi : \sum_{r=1}^{R} a_r \otimes b_r \to \sum_{r=1}^{R} \varphi(a_r)\psi(b_r) \tag{3.1}$$

is bounded, and extends to the product $A \otimes B$.

THEOREM 2. *If $a \in A^n$ and $b \in B^m$ are arbitrary then the system*

$$(a \otimes 1, 1 \otimes b) = (a_1 \otimes 1, a_2 \otimes 1, \ldots, a_n \otimes 1, 1 \otimes b_1, \ldots, 1 \otimes b_m)$$

has left spectrum given by the product

$$\sigma^{\text{left}}_{A \otimes B}(a \otimes 1, 1 \otimes b) = \sigma^{\text{left}}_A(a) \times \sigma^{\text{left}}_B(b). \tag{3.2}$$

Similarly for the right spectrum; for single elements $a = a_1 \in A$ and $b = b_1 \in B$ there is inclusion

$$\partial\,(\sigma_A(a) \times \sigma_B(b)) \subseteq \sigma^{\text{joint}}_{A \otimes B}(a \otimes 1, 1 \otimes b) \subseteq \sigma_A(a) \times \sigma_B(b). \tag{3.3}$$

PROOF. The left-hand side of (3.2) is obviously included in the right; if, conversely, $s \in C^n$ is in $\sigma^{\text{left}}_A(a)$ and $t \in C^m$ in $\sigma^{\text{left}}_B(b)$, then the systems $a - s$ and $b - t$ generate proper closed left ideals M and N in A and B. By the Hahn-Banach theorem there exist bounded linear functionals $\varphi \in A^*$ and $\psi \in B^*$ for which $\varphi(1) = \psi(1) = 1$, while $\varphi(M) = \psi(N) = \{0\}$. Now the functional $\varphi \otimes \psi$ of (3.1) annihilates the left ideal generated by the system $((a - s) \otimes 1, 1 \otimes (b - t))$ in the algebra $A \otimes B$, but not the identity $1 \otimes 1$. This puts $(s, t) \in C^{n+m}$ in the left spectrum of the system $(a \otimes 1, 1 \otimes b)$.

For the inclusion (3.3) we use the fact [**5**, Lemma 4.1] that the topological boundary of the spectrum of a single element in a Banach algebra lies in the intersection of its left and right spectra.

4. **Spectral mapping theorems.** The combination of (3.2) from Theorem 2 with (2.10) from Theorem 1 gives at once

THEOREM 3. *If $a \in A^n$ and $b \in B^m$ are commuting systems of elements, and $f \in \mathrm{Poly}^p_{m+n}$, then there is an equality*

$$\sigma^{\text{left}}_{A \otimes B} f(a \otimes 1, 1 \otimes b) = f(\sigma^{\text{left}}_A(a) \times \sigma^{\text{left}}_B(b)). \tag{4.1}$$

Similarly for right spectra; for single elements $a = a_1 \in A$ and $b = b_1 \in B$, and one polynomial in two variables $f = f_1 \in \mathrm{Poly}_2$, there is equality

$$\sigma_{A \otimes B} f(a \otimes 1, 1 \otimes b) = f(\sigma_A(a) \times \sigma_B(b)). \tag{4.2}$$

PROOF. For the second part apply (3.3), together with a simple observation about polynomials in two complex variables:

$$f(\partial(\sigma_A(a) \times \sigma_B(b))) = f(\sigma_A(a) \times \sigma_B(b)). \tag{4.3}$$

One way to see this is to count the zeroes of the polynomial $f(\cdot, w) - r$ in the interior of the compact set $\sigma_A(a)$, for each complex number r and each point w of $\sigma_B(b)$; compare Lemma 2.2 of [**12**].

The Brown-Pearcy result [**1**] is the case $f(a \otimes 1, 1 \otimes b) = a \otimes b$, with

$A = B = \mathscr{L}(E, E)$ for a Hilbert space E. Our arguments readily extend to Schechter's generalization [**12**], which covers the product of n copies of $A = \mathscr{L}(E, E)$ for a Banach space E, and rational functions f with no singularities on the joint spectrum. Note carefully the difference between the "joint spectrum" of Schechter's paper [**12**] and ours in (2.1).

If only one of the systems $a \in A^n$ and $b \in B^m$ is commutative we still, using (2.7) and (2.9) instead of (2.10), obtain a result sufficient to determine the spectrum of an "operator matrix":

THEOREM 4. *If $a \in A^n$ is a commuting system, if $b \in B^m$ is arbitrary, and if $f \in \mathrm{Poly}^p_{m+n}$ is a system of polynomials, then there is equality*

$$\sigma^{\mathrm{left}}_{A \otimes B} f(a \otimes 1, 1 \otimes b) = \bigcup \{\sigma^{\mathrm{left}}_B f(s, b) : s \in \sigma^{\mathrm{left}}_A(a)\}. \tag{4.4}$$

PROOF. The right-hand side of (4.4) is included in the left because, if $s \in C^n$ is in $\sigma^{\mathrm{left}}_A(a)$ and $r \in C^p$ in $\sigma^{\mathrm{left}}_B f(s, b)$, then by (3.2), the system $(s, r) \in C^{n+p}$ is in $\sigma^{\mathrm{left}}_{A \otimes B}(a \otimes 1, 1 \otimes f(s, b))$, and by (2.7), also in $\sigma^{\mathrm{left}}_{A \otimes B}(a \otimes 1, f(a \otimes 1, 1 \otimes b))$. Conversely if r is in the left-hand side of (4.4) we apply (2.9) to find $s \in C^n$ for which (s, r) is in $\sigma^{\mathrm{left}}_{A \otimes B}(a \otimes 1, f(a \otimes 1, 1 \otimes b))$, and use (2.7) again.

For the application to "operator matrices" take $B = C_{qq}$ to be the algebra of $q \times q$ complex matrices, so that the tensor product $A \otimes_C B$ is "$q \times q$ matrices with entries in A": All the uniform crossnorms give the same Cartesian product topology. If we take $b = (b_{11}, b_{12}, \ldots, b_{qq}) \in B^{q^2}$ to be the usual basis for the vector space B then an arbitrary matrix can be written

$$f(a \otimes 1, 1 \otimes b) = \sum_{j,k=1}^{q} a_{jk} \otimes b_{jk}\,; \tag{4.5}$$

we claim that, for a commuting system of entries $a = (a_{11}, a_{12}, \ldots, a_{qq})$,

$$\sigma_{A \otimes B} f(a \otimes 1, 1 \otimes b) = \{r \in C : 0 \in \sigma_A \det(f(a \otimes 1, 1 \otimes b) - rI)\}. \tag{4.6}$$

The result can be obtained [**9**, Chapter 5] by extending the numerical determinant theory: here we use (4.4) on the left-hand side of (4.6), and apply (2.4) to the right-hand side.

5. **Multiplication operators.** Associated with a system $a \in A^n$ of Banach algebra elements are the systems L_a and R_a of multiplication operators, where, for each $j = 1, 2, \ldots, n$,

$$L_{a_j}(x) = a_j x \quad (x \in A) \qquad \textit{and} \qquad R_{a_j}(x) = x a_j \quad (x \in A). \tag{5.1}$$

Lumer and Rosenblum obtained the analogue of (4.2), with L_a and R_b in place of $a \otimes 1$ and $1 \otimes b$, in the case $A = \mathscr{L}(E, E)$ for a Banach space E. To summarize a derivation of this result we recall the left and right "approximate point spectrum" [**5**, Definition 1.3] of a system of Banach algebra elements:

$$\tau_A^{\mathrm{left}}(a) = \left\{ s \in C^n : \inf_{\|x\| \geqq 1} \sum_{j=1}^{n} \|(a_j - s_j)x\| = 0 \right\} \tag{5.2}$$

and

$$\tau_A^{\mathrm{right}}(a) = \left\{ s \in C^n : \inf_{\|x\| \geqq 1} \sum_{j=1}^{n} \|x(a_j - s_j)\| = 0 \right\}. \tag{5.3}$$

Of course these are subsets of the left and right spectra (2.2) and (2.3); there is equality if $A = \mathscr{L}(E, E)$ is the bounded linear operators on a Hilbert space [**5**, Theorem 2.5], [**2**], and for a single element $a = a_1$ the topological boundary of the spectrum includes the intersection of (5.2) and (5.3) [**5**, Lemma 4.1]. The results of Lumer and Rosenblum [**10**] can be derived from

THEOREM 5. *If $A = \mathscr{L}(E, E)$ for a Banach space E, and if $S \in A^n$ and $T \in A^m$ are systems of bounded linear operators, then there is inclusion*

$$\tau_A^{\mathrm{left}}(S) \times \tau_A^{\mathrm{right}}(T) \subseteq \sigma_{\mathscr{L}(A,A)}^{\mathrm{left}}(L_S, R_T) \subseteq \sigma_A^{\mathrm{left}}(S) \times \sigma_A^{\mathrm{right}}(T) \tag{5.4}$$

and

$$\tau_A^{\mathrm{right}}(S) \times \tau_A^{\mathrm{left}}(T) \subseteq \sigma_{\mathscr{L}(A,A)}^{\mathrm{right}}(L_S, R_T) \subseteq \sigma_A^{\mathrm{right}}(S) \times \sigma_A^{\mathrm{left}}(T). \tag{5.5}$$

For single operators $S = S_1$ and $T = T_1$ there is inclusion

$$\partial\, (\sigma_A(S) \times \sigma_A(T)) \subseteq \sigma_{\mathscr{L}(A,A)}^{\mathrm{joint}}(L_S, R_T) \subseteq \sigma_A(S) \times \sigma_A(T). \tag{5.6}$$

PROOF. The arguments for (5.4) and (5.5) are extracted from the proofs of Theorem 9 and Theorem 10 of Lumer and Rosenblum [**10**]; then (5.6) follows in the same way as (3.3).

For one polynomial $f = f_1$ in two variables, and for operators $S = S_1$ and $T = T_1$ on a Banach space it follows, analogous to (4.3), that

$$\sigma_{\mathscr{L}(A,A)} f(L_S, R_T) = f(\sigma_A(S) \times \sigma_A(T)). \tag{5.7}$$

This of course is the result of Lumer and Rosenblum [**10**, Theorem 10]. Also for a Hilbert space E we obtain equality throughout (5.4) and (5.5), and hence analogues for Theorems 3 and 4.

REFERENCES

1. A. Brown and C. Pearcy, *Spectra of tensor products of operators*, Proc. Amer. Math. Soc. **17** (1966), 162–169. MR **32** #6218.

2. L. A. Coburn and M. Schechter, *Joint spectra and interpolation of operators*, J. Functional Analysis **2** (1968), 226–237. MR **37** #3364.

3. J. Gil de Lamadrid, *Uniform cross norms and tensor products of Banach algebras*, Duke Math. J. **32** (1965), 359–368. MR **32** #8125.

4. P. R. Halmos, *A Hilbert space problem book*, Van Nostrand, Princeton, N.J., 1967. MR **34** #8178.

5. R. E. Harte, *Spectral mapping theorems*, Proc. Roy. Irish Acad. **72A** (1972), 89–107.
6. ———, *The spectral mapping theorem in several variables*, Bull. Amer. Math. Soc. **78** (1972), 870–874.
7. ———, *The spectral mapping theorem for quasicommuting systems*, Proc. Roy. Irish Acad. **73A** (1973), 7–18.
8. ———, *Tensor products, multiplication operators and the spectral mapping theorem*, Proc. Roy. Irish Acad. (to appear).
9. K. Hoffman and R. Kunze, *Linear algebra*, Prentice-Hall Math. Ser., Prentice-Hall, Englewood Cliffs, N.J., 1961. MR **23** #A3146.
10. G. Lumer and M. Rosenblum, *Linear operator equations*, Proc. Amer. Math. Soc. **10** (1959), 32–41. MR **21** #2927.
11. R. Schatten, *A theory of cross-spaces*, Ann. of Math. Studies, no. 26, Princeton Univ. Press, Princeton, N.J., 1950. MR **12**, 186.
12. M. Schechter, *On the spectra of operators on tensor products*, J. Functional Analysis **4** (1970), 95–99.

Department of Mathematics, University College, Cork, Ireland