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FREE FINITE GROUP ACTIONS ON S^3

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In this paper we describe the first stages of a theory of 3-manifolds with finite fundamental group. The strong conjecture that any free finite group action on S^3 is conjugate to a linear action is known for some cyclic groups, see [**3**], [**4**], and is supported by recent work of one of us on fundamental groups [**2**]. Here we concern ourselves with the weaker conjecture that any compact 3-manifold with finite fundamental group is homotopy equivalent to a Clifford-Klein form. (Note that both conjectures are phrased to avoid problems with homotopy 3-spheres.) It is known, see for example [**6**], that the homotopy type of such a manifold is determined by the fundamental group and the first k-invariant. By exploiting the link between k-invariant and finiteness obstruction we are able to decide which homotopy types correspond to *finite* Poincaré complexes, and thus restrict the possible homotopy types for manifolds. There are nonstandard types for some groups, and a corollary of our argument is the existence in dimensions $4n - 1$, $n \geqq 2$, of free actions homotopically distinct from orthogonal ones. When $n = 1$, we can only produce such an action on a homology sphere, and it would be most interesting to know the fundamental group.

1. **Homotopy type of space forms.** Let the abstract group π be isomorphic to the fundamental group of a compact 3-dimensional manifold of constant positive curvature (Clifford-Klein form), and suppose π cannot be decomposed as a direct product. The possibilities for π are listed in the following table, see [**10**, Chapter 7, p. 224]:

If Y is a 3-dimensional CW-complex such that $\tilde{Y}$ is homotopy equivalent to S^3, and we can choose an isomorphism $\psi:\pi_1(Y, y) \to \pi$, we shall call Y a *Poincaré space form*. Y is not necessarily finite, and the isomorphism ψ, although not natural, is assumed fixed. Homotopy classes of space forms are in (1-1) correspondence with the orbits in $H^4(\pi, \boldsymbol{Z})$ under the action of $\pm \mathrm{Aut}\, \pi$ [**6**, Theorem 1.8] and there is a well-defined obstruction to finding a finite complex in a given homotopy type, lying in the projective class group $\tilde{K}_0(\boldsymbol{Z}\pi)$ [**8**, Theorem F]. We can describe this obstruc-

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π	Generators	Relations	Order
Z_m	A	$A^m = 1$	m
Q_{4n}	A, B	$A^{2n} = 1, A^n = B^2, B^{-1}AB = A^{-1}$	$4n$
T_v^*	A, P, Q	$A^{3^v} = 1, P^4 = 1, P^2 = Q^2,$ $PQP^{-1} = Q^{-1}, APA^{-1} = Q, AQA^{-1} = PQ$	8.3^v, $v \geqq 1$
0^*	A, P, Q, R	As for T_1^*, also $R^2 = P^2$, $RPR^{-1} = QP, RQR^{-1} = Q^{-1}, RAR^{-1} = A^{-1}$	48
I^*	A, B, C	$A^6 = 1, A^3 = B^2 = C^5 = ABC$	120

TABLE 1

tion explicitly in the following way. Since π is isomorphic to the fundamental group of some space form, the chains of the universal cover define an exact sequence of finitely generated π-modules

$$0 \to Z \to F_3 \to F_2 \to F_1 \to F_0 \to Z \to 0,$$

whose chain homotopy type corresponds to a generator k_0 of $H^4(\pi, Z)$. If r is any integer prime to $[\pi : 1]$, and $\Sigma = \sum_{g \in \pi} g$, let $[r, \Sigma]$ be the projective ideal of $Z\pi$ generated by r and Σ. There is an isomorphism, see [**5**, Lemma 6.1],

$$Z + Z[\pi] \xrightarrow{\cong} Z + [r, \Sigma],$$

given by

$$(1, 0) \mapsto (\bar{r}, m\Sigma) \quad \& \quad (0, 1) \mapsto (1, r), \qquad r\bar{r} = 1 + m[\pi : 1].$$

It is easy to construct a map of degree $\bar{r}\,(\mathrm{mod}[\pi : 1])$ between the chain complex above and the modified complex

$$0 \to Z \to F_3 + Z\pi \to F_2 + [r, \Sigma] \to F_1 \to F_0 \to Z \to 0,$$

a geometric realisation of which corresponds to the generator $\bar{r}k_0$ of $H^4(\pi, Z)$. Since $[\bar{r}, \Sigma] + [r, \Sigma]$ is free, this shows that $[\bar{r}, \Sigma]$ is the finiteness obstruction for this homotopy type.

Divide the homotopy classes of Poincaré space forms Y (equivalently orbits in $H^4(\pi, Z)$) into three types:

I, Y is infinite, hence not equivalent to a manifold,

F, Y is finite, and

CK, Y is equivalent to one of the classical Clifford-Klein forms.

The argument above shows that we may distinguish between types I and F by means of the modules $[r, \Sigma]$; the orbit of rk_0 belongs to F if and only if $[r, \Sigma] \sim 0$ in $\tilde{K}_0(\mathbf{Z}\pi)$.

THEOREM 1. *If the 2-Sylow subgroups of π are cyclic, $[r, \Sigma] \sim 0$ for all r. If the 2-Sylow subgroups of π are generalised quaternion, $[r, \Sigma] \sim 0$ provided $r \equiv \pm 1(8)$.*

PROOF. We outline the argument for cyclic and quaternion groups, since these are typical. Observe that the module $[r, \Sigma] \sim 0$ if there is an element in the orbit of rk_0 coming from a form defined by a fixed point free representation of π. For convenience we identify $H^4(\pi, \mathbf{Z})$ with $\mathbf{Z}_{[\pi:1]}$ by mapping the generator k_0 to 1. The symbol φ denotes Euler's function, and U the group of units.

$\mathbf{Z}_m$: $[r, \Sigma] \sim 0$, since the homotopy type contains the lens space $L(m, \bar{r})$.

Q_{4n}: (i) $n = 2$. If $H = \mathbf{Z}[i, j, k] \subseteq U(S^3)$, one looks at the exact sequence

$$\to K_1(H) + U(\mathbf{Z}(2\mathbf{Z}_2)) \to U(F_2(2\mathbf{Z}_2)) \to \tilde{K}_0(\Lambda) \to,$$

and sees that $[3, \Sigma]$ comes from a unit in $F_2(2\mathbf{Z}_2)$, which is not hit by any element in $K_1(H)$. This computation was originally performed by one of us and C. T. C. Wall.

(ii) $n = 2t$, and $Q_{4n} \supseteq Q_8$. If $r \equiv \pm 3(8)$ the orbit of r lifts to the orbit of 3 in $\mathbf{Z}_8$, which shows that the covering complex with fundamental group Q_8 cannot be finite. The remaining modules are the trivial finiteness obstructions associated with the CK-forms. Each of these is defined by a fixed point free representation corresponding to minus a square in $U(\mathbf{Z}_{8t})$.

(iii) $n = 2t + 1$. As in (i) we decompose Λ to show $[r, \Sigma] \sim 0$ always; hence for these groups, there are "bad" finite complexes.

We present the geometric implications of our theorem in Table 2 below. Note that the figures refer to numbers of generators rather than to number of orbits under the $\pm \mathrm{Aut}\, \pi$ action.

2. **Homotopically exotic actions.** The argument of the previous paragraph implies the existence of finite Poincaré complexes homotopically distinct from manifolds of constant positive curvature in all dimensions $\equiv 3(4)$. Furthermore the Spivak normal fibration admits a PL-bundle structure in all cases, see [7], and the problem of finding a PL-manifold in the homotopy type reduces to one of surgery. The following theorem solves this problem for certain metacyclic groups and shows the existence of homotopically exotic, free, piecewise linear actions on spheres.

THEOREM 2. *Let n be an odd prime and Q_{4n} the generalised quaternion group of order $4n$. Then, for every integer $l > 0$, there exists a PL, closed, oriented $(4l + 3)$-manifold M with the following properties:*

(i) $\pi_1 M = Q_{4n}$,

π	$\varphi[\pi:1]$	I	CK	F but not CK
$\boldsymbol{Z}_m$	$\varphi(m)$	—	$\varphi(m)$	—
Q_{4n}: $n = 2^k(2t+1)$, $k \geqq 1$	$2^{k-1}\varphi(2t+1)$	$2^{k-2}\varphi(2t+1)$	$2^{k-2}\varphi(2t+1)$	—
Q_{4n}: $n \equiv 1(2)$	$2\varphi(n)$	—	$\varphi(n)$	$\varphi(n)$
$T_1^*, 0^*, I^*$	8,16, 32	4, 8, 16	2, 4, 4	
$T_v^*, v \geqq 2$	8.3^{v-1}	4.3^{v-1}	2.3^{v-1}	

TABLE 2

(ii) *the universal cover* $\tilde{M}$ *is* S^{4l+3}, *and*

(iii) $\pm$ *k-invariant of M is not a square modulo* $4n$.

If $l = 0$, *there exists a free action of* Q_{4n} *on a homology* 3-*sphere, such that the k-invariant of the action satisfies* (iii).

PROOF. Let $D_{2n} = \{A, B: A^n = B^2 = 1, BAB^{-1} = A^{-1}\}$ be the dihedral group of order $2n$, and let $\boldsymbol{Z}[e^{2\pi i/n}, j]$ be the subring of the quaternion numbers generated by $e^{2\pi i/n}$ and j. There is a cube of ring homomorphisms (shown in collapsed form),

$$\begin{tikzcd}
\boldsymbol{Z}[Q_{4n}] \arrow[r, "\langle B^2-1\rangle"] \arrow[d, "{\langle 1-A+A^2-\cdots+A^{n-1}\rangle}"'] & \boldsymbol{Z}[D_{2n}] \arrow[r, "\langle A^2-1\rangle"] \arrow[d, "{\langle 2, 1+A+\cdots+A^{n-1}\rangle}"'] & \boldsymbol{Z}[\boldsymbol{Z}_2] \arrow[d] & \boldsymbol{Z}[\boldsymbol{Z}_4] \arrow[l, "\langle B^2-1\rangle"'] \arrow[d, "{\langle n, B^2+1\rangle}"'] & \boldsymbol{Z}[Q_{4n}] \arrow[l, "\langle A^2-1\rangle"'] \arrow[d, "{\langle 1-A+A^2-\cdots+A^{n-1}\rangle}"'] \\
\boldsymbol{Z}[e^{2\pi i/n}, j] \arrow[r, "\langle 2\rangle"] & M(2;\boldsymbol{F}_2[\zeta+1/\zeta]) \arrow[r] & 0 & \boldsymbol{F}_n[j] \arrow[l] & \boldsymbol{Z}[e^{2\pi i/n}, j] \arrow[l, "\langle e^{2\pi i/n}-1\rangle"']
\end{tikzcd}$$

which leads to an exact sequence of Wall groups:

$$0 \to L_3(\boldsymbol{Z}[Q_{4n}]) \xrightarrow{\mathscr{X}} L_3(\boldsymbol{Z}[\boldsymbol{Z}_4]) \oplus L_3(\boldsymbol{Z}[D_{2n}]) \oplus L_3(\boldsymbol{Z}[e^{2\pi i/n}, j])$$

$$\to L_3(\boldsymbol{F}_n[j]) \oplus L_3(\boldsymbol{Z}[\boldsymbol{Z}_2]) \oplus L_3(M(2;\boldsymbol{F}_2[\zeta+1/\zeta])) \to \cdots.$$

We look at the image of our surgery obstruction under $\mathscr{X}$. According to H. Bass, I. Berstein and others, the Wall group of $\boldsymbol{Z}_4$ is cyclic of order 2, and by the calculation in [**1**], an element in $L_3(\boldsymbol{Z}[D_{2n}])$ is detected by its image in $L_3(\boldsymbol{Z}[\boldsymbol{Z}_2])$ and semicharacteristic classes. The third group $L_3(\boldsymbol{Z}[e^{2\pi i/n}, j])$ can be studied through the homomorphism

$$\text{Spin}: L_3(\boldsymbol{Z}[e^{2\pi i/n}, j]) \to \boldsymbol{Q}[2\cos(2\pi/n)]^{\times} \Big/ \begin{array}{l}\text{modulo subgroup generated}\\ \text{by units and squares in}\\ \boldsymbol{Z}[2\cos(2\pi/n)]\end{array}$$

defined by the spinor norm, see [**9**]. Spin is actually injective, and for the problem on hand, one can show that both the spinor norm and the semicharacteristic classes are zero. Possibly by changing the normal invariant we can kill off any obstruction in $L_3(\mathbf{Z}[\mathbf{Z}_4])$, and by exactness the theorem follows.

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