ORBITS OF FAMILIES OF VECTOR FIELDS
AND INTEGRABILITY OF SYSTEMS
WITH SINGULARITIES

BY HECTOR J. SUSSMANN

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Let X be a C^∞ vector field on the C^∞ manifold M. We use $t \to X_t(m)$ to denote the integral curve of X which passes through m when $t = 0$. Let D be a set of C^∞ vector fields on M. Two points m and m' of M are said to be D-*connected* if there exist elements $X^1, \ldots, X^k$ of D and real numbers $t_1, \ldots, t_k$ such that

$$m' = X^1_{t_1}(X^2_{t_2}(\cdots X^k_{t_k}(m) \cdots)).$$

This defines an equivalence relation on M. The equivalence classes are called the *orbits* of D.

Let S be an orbit of D. For each $m \in S$ and each finite sequence $\xi = (X^1, \ldots, X^k)$ of elements of D, let $F_{\xi,m}$ denote the map

$$(t_1, \ldots, t_k) \to X^1_{t_1}(X^2_{t_2}(\cdots X^k_{t_k}(m) \cdots)).$$

It is clear that $F_{\xi,m}$ is a C^∞ mapping from an open subset U of R^k into M. Moreover the range of $F_{\xi,m}$ is a subset of S. We topologize S by the strongest topology for which all the maps $F_{\xi,m}$ are continuous.

THEOREM 1. *S is a connected C^∞ submanifold of M.*

A *distribution* on M is a mapping H which assigns to every $m \in M$ a linear subspace $H(m)$ of the tangent space of M at m. It is not required that the dimension of $H(m)$ be constant. A vector field X defined in an open subset U of M *belongs* to the distribution H if $X(m) \in H(m)$ for every $m \in U$. We say that H is a C^∞ *distribution* if, for every $m \in M$ and every $v \in H(m)$, there exists a C^∞ vector field X such that X belongs to H and $X(m) = v$. If D is a set of vector fields and H is a distribution, we say that H is D-invariant if, whenever $m \in M$, $X \in D$, and t is a real number such that $X_t(m)$ is defined, it follows that the differential of X_t maps $H(m)$ into $H(X_t(m))$. Given a set D of C^∞ vector fields on M, there exists a smallest distribution H which is D-invariant and is such that every element of D belongs to H. Let this distribution be denoted by P_D. Then P_D is a C^∞ distribution.

Integral manifolds and maximal integral manifolds of a distribution

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are defined in the usual way. In the statement of the following theorem, the orbits of D are given the topology that was defined in the remarks preceding Theorem 1, and the differentiable structure whose existence is asserted in Theorem 1. This structure is clearly unique.

THEOREM 2. *The orbits of D are maximal integral manifolds of P_D.*

The preceding result shows that the distribution P_D is *integrable*, in the sense that through every point of M there passes a maximal integral manifold of P_D. It turns out that the converse is also true: Every integrable distribution is of the form P_D for some D.

THEOREM 3. *Let H be a C^∞ distribution and let D be a set of C^∞ vector fields such that, for every $m \in M$, $H(m)$ is the linear hull of the vectors $X(m)$, $X \in D$. Then the following conditions are equivalent:*

(a) *Through every point of M there passes an integral manifold of H.*

(b) *Through every point of M there passes a maximal integral manifold of H.*

(c) $H = P_D$.

(d) *For every $m \in M$ there exist elements $X^1, \ldots, X^k$ of D such that $H(m)$ is the linear hull of $X^1(m), \ldots, X^k(m)$ and that the following holds: For every $X \in D$ there exist $\varepsilon > 0$ and C^∞ functions*

$$f^i_j : (-\varepsilon, \varepsilon) \to R \qquad (1 \le i, j \le k)$$

such that, for $-\varepsilon < t < \varepsilon$,

$$[X, X^i](X_t(m)) = \sum_{i=1}^{k} f^i_j(t) X^j(X_t(m)).$$

Proofs of our three theorems will appear elsewhere. We now discuss the connection of our results with those of Chow [1], Hermann [2], Lobry [3], Nagano [5] and Matsuda [4]. Let D^* be the smallest set of vector fields which contains all the elements of D and which is closed under Lie brackets. Let $I_D(m)$ denote the linear hull of the vectors $X(m)$, $X \in D^*$. Chow proved that the orbits of D are precisely the connected components of M, provided $I_D(m)$ has maximal dimension for each m. If the assumption of Chow's theorem is satisfied, it is clear that $I_D(m) = P_D(m) =$ tangent space of M at m. Therefore Chow's result is a particular case of Theorem 2. Hermann and Lobry studied (under the name of "leafs") the orbits of D under the assumption that the distribution I_D is integrable. By applying Chow's theorem to the integral manifolds of I_D, they concluded that the orbits of D are the maximal integral manifolds of I_D. Theorem 1 shows that the orbits of D are always smooth submanifolds of M, and that the integrability of I_D is not needed. Theorem 2 shows that the orbits are the maximal integral manifolds of an integrable distribution P_D. In general,

P_D will not coincide with I_D and, therefore, I_D is not the "correct" distribution to look at.

The integrability of smooth distributions H which are *involutive* (i.e., whenever X and Y belong to H then $[X, Y]$ belongs to H) is a classical result if the dimension of $H(m)$ is constant. In general (i.e., when H has "singularities") the condition that H be involutive is necessary but not sufficient. Sufficient conditions were given by Hermann [2] (H is "locally finitely generated"), Lobry [3] (H is "locally of finite type") and Matsuda [4] (H satisfies a "convergence condition"). It is easy to see that Lobry's condition (which is weaker than Hermann's) implies condition (d) of Theorem 3. Also, Matsuda's condition implies (c). Therefore, these results are all contained in Theorem 3.

Nagano [5] proved the result (stated by Hermann in [2]) that every analytic involutive distribution on a real analytic manifold M is integrable. This also follows from Theorem 3, because in the analytic case condition (d) is always satisfied (cf. Lobry [3]).

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CHICAGO, CHICAGO, ILLINOIS 60637

DEPARTMENT OF MATHEMATICS, RUTGERS UNIVERSITY, NEW BRUNSWICK, NEW JERSEY 08903 (Current address)