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HIGHER K-THEORY FOR REGULAR SCHEMES

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ABSTRACT. Higher K-groups are defined for regular schemes, generalizing the K-theory of Karoubi and Villamayor. A spectral sequence is developed which shows how the K-groups depend on the local rings of the scheme. Applications to curves and affine surfaces are given.

Let X be a regular separated scheme. If U is an affine open subset of X, then the assignment $U \mapsto \mathrm{BGl}(S^n\Gamma(U, O_X)_*)$ is a sheaf of Kan complexes on the Zariski site. Here S denotes the suspension ring functor of Karoubi [**10**] and if A is a ring, A_* denotes the simplicial ring [**11**]

$$(A_*)_n = A[t_0, t_1, \ldots, t_n]/(t_0 + \cdots + t_n - 1).$$

We recall that $\pi_i \mathrm{BGl} A_* = K^{-i}A$, $i \geqq 1$ [**11**], where the K-groups of Karoubi and Villamayor are indicated [**10**]. Also, recall that $K_0(A) \times \mathrm{BGl}(A_*) \simeq \Omega \mathrm{BGl}(SA_*)$ if A is K-regular ([**9**], [**8**]). Thus there is a sheaf of Kan spectra $E(O_X)$ on X associated to the pre-spectrum $U \mapsto (n \mapsto \mathrm{BGl}(S^n\Gamma(U, O_X)_*))$. Such sheaves have been studied by K. Brown [**4**] who has defined cohomology with coefficients in a sheaf of Kan spectra: $H^n(X, E(O_X))$, $n \in Z$.

DEFINITION. $K^n(X) = H^n(X, E(O_X))$.

We remark that the spectra $E(O_X)$ are connected since X is regular, so $K^i(X) = 0$ if $i > 0$. The main properties of these groups and most of the motivation for introducing them are summarized in

THEOREM 1. *Let X be a regular separated scheme.*

(1) *If U and V are open subschemes of X, then there is an exact Mayer-Vietoris sequence*

$$\cdots \to K^{i-1}(U \cap V) \to K^i(U \cup V) \to K^i(U) \oplus K^i(V) \to K^i(U \cap V) \to \cdots$$

(2) *If X has finite (Krull) dimension, then there is a fourth quadrant spectral sequence of cohomological type*

$$E_2^{pq} = H^p(X, \underline{K}^q) \Rightarrow K^{p+q}(X).$$

Here $\underline{K}^q$ is the sheaf in the Zariski site associated to the presheaf

$$U \mapsto K^q(\Gamma(U, O_X)), \qquad U \text{ affine open.}$$

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(3) *If* $X = \operatorname{spec} A$, *then* $K^i(X) = K^i(A)$, *the Karoubi-Villamayor K-groups of* A.

(4) $K^i(X \times_{\operatorname{spec} \mathbf{Z}} \operatorname{spec} \mathbf{Z}[t]) = K^i(X)$.

The properties (1) and (2) are formal properties of sheaves of Kan spectra [4]. Property (4) is immediate, since the Karoubi-Villamayor theory is invariant under polynomial extension. Property (3) however is a theorem whose proof depends on the main results of [7]. In addition the Krull theory of divisors enters in the description of the affine opens of spec A. Full details will be published elsewhere.

Properties (1) and (3) actually serve to provide an axiomatic characterization for the theory, in the category of regular separated noetherian schemes, as a simple induction argument shows. Also, since $K^n(X)$ arise as the homotopy of a spectrum, these groups are the coefficient groups (cohomology of a point) in a generalized cohomology theory of complexes.

COROLLARY 2. *If X is a regular curve, then there are short exact sequences*

$$0 \to H^1(X, \underline{K}^{-n-1}) \to K^{-n}(X) \to H^0(X, \underline{K}^{-n}) \to 0, \quad \textit{all } n \geqq 0.$$

This is merely the fact that the spectral sequence of Theorem 1 degenerates at the E_2 level for curves.

PROPOSITION 3. *If A is a Dedekind ring with field of fractions F, then the sequence*

$$K_2(A) \to K_2(F) \to \coprod_{\underline{m} \in \max A} K_1(A/\underline{m}) \to K_1(A) \to \cdots$$

is exact.

REMARK. Exactness at K_2F was shown by Bass if A has only countably many maximal ideals [2]. Exactness at other points is classical.

One makes use of the recently discovered short exact sequence of K. Dennis and M. R. Stein [6] to construct a short exact sequence of sheaves

$$0 \to \underline{K^{-2}} \to \underline{K_2 F_X} \to \coprod_v K_1(k_v) \to 0.$$

Here $\underline{K_2 F_X}$ is the constant sheaf where F is the field of rational functions of X, and $\coprod_v K_1(k_v)$ assigns to each open set U the group

$$\textstyle\coprod_{v \in U;\, v \text{ closed}} K_1(k_v),$$

where k_v is the residue class field at v. One takes the long exact cohomology sequence associated to this short exact sequence of sheaves, and splices it to the short exact sequences of Corollary 2 to get the result.

PROPOSITION 4. *Let X be a regular affine surface and suppose that ξ is a vector bundle on X. Suppose in addition that* $\det \xi$, *the determinant bundle in* Pic X, *is trivial. Then ξ and rank ξ have the same class in $K^0(X)$ if and only if $c_2(\xi) = 0$, where $c_2(\xi) \in H^2(X, \underline{K}^{-2})$ is the universal second Chern class of ξ.*

The interpretation of the class of ξ in $H^2(X, \underline{K}^{-2})$ as a universal Chern class is suggested by recent work of Spencer Bloch. This result follows from the spectral sequence of Theorem 1 with the observation that the differential $d_2: H^0(X, \underline{K}^{-1}) \to H^2(X, \underline{K}^{-2})$ is zero, since $H^0(X, \underline{K}^{-1}) = \Gamma(X, O_X^*) = U(A)$ is a direct factor of $K^{-1}(X) = K_1(A)$, where $A = \Gamma(X, O_X)$.

Denote now by $K_i^{ab}(X), i = 0, 1$, the K-groups of the abelian category of coherent O_X modules [**1**]. There are natural morphisms $K_i^{ab} \to K^{-i}$. Of course, by Theorem 1, $K_i^{ab}(X) = K^{-i}(X)$ if X is affine and regular ($i = 0, 1$). From the Mayer-Vietoris sequence of Theorem 1 and the corresponding Mayer-Vietoris sequence for K_0^{ab} (which can be deduced from [**5**, Proposition 7]), it follows that if the regular scheme X is the union of two open affines, then $K_0^{ab}(X) = K^0(X)$. In particular this holds for curves. However, we do not know how much more generally this result holds.

Concerning K_1^{ab} the result is less satisfactory. Using the results of L. Robert's thesis [**12**] we can show

PROPOSITION 5. *If X is a complete nonsingular elliptic curve over the complex numbers, then $K_1^{ab}(X) \to K^{-1}(X)$ is surjective but not injective.*

If X is a complete nonsingular curve over the constant field k, the algebraic closure of a finite field, then using results of Tate [**13**] we can show that $K^{-2}(X) = \mathrm{Tor}(k^*, \mathrm{Pic}(X))$ and $K^{-1}(X) = k^* \otimes \mathrm{Pic}\, X \cong k^*$. The first assertion amounts to an identification of $K^{-2}(X)$ with the tame kernel in the function field case.

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