

DISSECTIONS OF A SIMPLEX

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Communicated by Paul T. Bateman, June 29, 1972

1. On each edge e_i of a tetrahedron T , let $n_i \geq 0$ points be selected and each joined by a triangular plane segment to the corresponding opposite edge of T . In [1] we proved that the number C of disjoint cells formed inside T satisfies

$$C \leq 1 + \sigma_1 + \sigma_2 + \sigma_3^*,$$

where σ_1 is the sum of the six n_i , σ_2 is the sum of the 15 products $n_i n_j$ with $1 \leq i < j \leq 6$, and σ_3^* is the sum of the 16 products $n_i n_j n_k$ for which the corresponding edges do not form a triangular path on T and $1 \leq i < j < k \leq 6$. The equality holds if and only if the triangular plane segments are in "interior general position," i.e., no three meet in an interior line segment and no four meet in an interior point of T .

In this note we announce the analogous result for a dissected simplex in E^d .

2. Let T^d be a simplex in E^d , the convex hull of $d + 1$ independent vertices V_0, V_1, \dots, V_d , and denote by e_{ij} the edge of T^d that joins V_i and V_j . On each edge e_{ij} let $n_{ij} \geq 0$ points (not vertices) be selected. With the $d - 1$ opposite vertices of T^d , each such point determines a $(d - 1)$ -simplex, which dissects T^d . Let H be the number of nonoverlapping $(d - 1)$ -polytopes into which the $\sum n_{ij}$ dissecting simplices divide T^d .

For each $m = 1, 2, \dots, d$, let

$$\sigma_m^* = \sum^* n_{i_1 j_1} n_{i_2 j_2} \cdots n_{i_m j_m},$$

the sum being extended over all m -sets of edges $e_{i_1 j_1}, e_{i_2 j_2}, \dots, e_{i_m j_m}$ no subcollection of which forms a closed path on T^d ; and let $H_d = 1 + \sum_{m=1}^d \sigma_m^*$.

Our result is that $H \leq H_d$, and the equality holds if and only if the dissecting simplices are in "interior general position," i.e., for each k with $2 \leq k \leq d$, no $k + 1$ dissecting simplices meet in a $(d - k)$ -simplex.

The proof is by induction on the dimension d . Let $\mathbf{n} = \langle n_{ij} \rangle$ be the $\binom{d+1}{2}$ -vector of data for T^d , the components being listed in lexicographic order. Let T_b be the "base" $(d - 1)$ -simplex, determined by the vertices V_0, V_1, \dots, V_{d-1} , and let \mathbf{n}_b be the $\binom{d}{2}$ -vector of data for T_b , with the components in lexicographic order. Let s_{ij} be the $\binom{d}{2}$ -vector having 1 in the

AMS (MOS) subject classifications (1970). Primary 50B30.

ij th position and 0 elsewhere.

We show on geometric grounds that the maximum number H_d of non-overlapping $(d - 1)$ -polytopes formed in T^d satisfies the recursion

$$H_1(\langle n_{01} \rangle) = 1 + n_{01},$$

$$H_d(\mathbf{n}) = H_{d-1}(\mathbf{n}_b) + n_{0d}H_{d-1}(\mathbf{n}_b) + \sum_{m=1}^{d-1} n_{md}H_{d-1}\left(\mathbf{n}_b + \sum_{j=0}^{m-1} n_{jd}\mathbf{s}_{jm}\right)$$

and then verify that the sequence $\{H_d\}$ defined above is the only solution of this recursion. Details and some related results will be published elsewhere.

REFERENCE

1. G. L. Alexanderson and John E. Wetzel, *Dissections of a tetrahedron*, J. Combinatorial Theory **11** (1971), 58–66.

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