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COMBINATORIAL SYMMETRIES OF THE m-DISC. I

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In 1938–1939 P. A. Smith published this theorem about combinatorial symmetries of the m-disc having prime period p: If K denotes the fixed point set of $Z_p \times D^m \to D^m$, then K is a Z_p-homology manifold and Z_p-homology disc (see [**4**]).

The main result in this paper states that for odd primes p, there is a homogeneity property of the fixed point set K which is not implied by P. A. Smith's theorem: There is a characteristic class $h^p_{4*+K-1} \in H_{4*+K-1}(K/\partial K, Z_2)$, which must vanish if K is to be the fixed point set for some $Z_p \times D^m \to D^m$, where K denotes both the polyhedron K and its dimension. By homogeneity it is meant that the class vanishes if K is a PL manifold (see Theorem 1.1b). Theorem 1.2 is intended to clarify the mechanics that define h^p_{4*+K-1}: represent homology classes by singular Z_p-homology manifolds P_i; compute an invariant from the mid-dimensional intersection forms of the P_i; use the universal coefficient theorem to get h^p_{4*+K-1}. This is a procedure well known to workers in the field (see [**3**], [**6**]).

A key step in determining the properties of this characteristic class, relates an exponent 4 invariant of the fixed point set for $Z_p \times M \to M$, to the Z_p-index of M, where $Z_p \times M \to M$ is a combinatorial symmetry on a closed PL manifold M.

Results are stated only for primes of the form $p = 4q + 1$ with $q =$ odd. Similar results hold for other odd primes, but tables and invariants must be slightly modified.

In part, this is a correction to [**1**]. There, in a remark, it is said that the converse to P. A. Smith's theorem is true for odd primes, provided the potential fixed point set admits a "2-parameter cross section." This is not true, as Theorems 1.1, 1.3 below show.

1. **Characteristic classes measuring nonhomogeneity.** $(K, \partial K)$ denotes a Z_p-homology manifold pair, σ denotes any exponent 4 invariant that can be additively associated to quadratic forms over the integers having determinant prime to p; e.g., various combinations of Hasse symbol and discriminant type invariants; reduction mod p invariants.

THEOREM 1.1. *There is a characteristic class* $h^\sigma_{4*+K-1} \in H_{4*+K-1}(K/\partial K, Z_4)$, *satisfying*

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(a) h^{σ}_{4*+K-1} *is an intrinsic invariant for K, depending only on the PL homeomorphism type of K and the quadratic form invariant* σ.

(b) $h^{\sigma}_{4*+K-1} = 0$ *if K is an integral-homology manifold.*

(c) $\exists K_\sigma$ *which is a* Z_p*-homology disc, for which* $h_{4*+K_\sigma-1 \neq 0}$.

The geometric essence of h^{σ}_{4*+K-1} is this:

THEOREM 1.2. *For l large enough there exists in the interior of* $K \times D^l$ *a finite set of polyhedra* $\{P_i\}$, *satisfying*

(a) *each* P_i *has a linear disc bundle for regular neighborhood in* $K \times D^l$;

(b) *any characteristic class* h^{σ}_{4*+K-1} *can be computed from the* $\{\sigma(P_i)\}$, *e.g.,* $\sigma(P_i) = 0 \ \forall_i \Leftrightarrow h^{\sigma}_{4*+K-1} = 0$. *Here* $\sigma(P_i)$ *is the evaluation of* σ *on the mid-dimensional intersection form of* P_i.

If σ is taken to be the mod p discriminant of a quadratic form—this lies in the group of units of Z_p modulo the subgroup of square units—then h^{σ}_{4*+K-1} will be the image of a unique class in $H_{4*+K-1}(K/\partial K, Z_2)$. This latter class is denoted by h^{p}_{4*+K-1}.

THEOREM 1.3. *Suppose* $p = 4q + 1$ *with* $q = odd$. *If K is the fixed point set for a combinatorial symmetry* $\boldsymbol{Z}_p \times N \to N$ *defined on a PL manifold N, then* $h^{p}_{4*+K-1} = 0$.

2. **A PL equivariant index theorem.** $\boldsymbol{Z}_p \times M \to M$ denotes a combinatorial symmetry defined on the closed PL manifold M of dimension $4m$, having odd prime order. K denotes the fixed point set for $\boldsymbol{Z}_p \times M \to M$. λ_K denotes the mid-dimensional intersection form of K; det(K) denotes the mod p discriminant of λ_K; $r(K)$ denotes the rank of λ_K mod 2. $H_{2m}(M, Q)_\eta$ is the subspace of $H_{2k}(M, Q)$ killed by the norm $\eta \equiv 1 + t + t^2 + \dots + t^{p-1}$, where t is a multiplicative generator for $\boldsymbol{Z}_p$. λ_η is the restriction to $H_{2m}(M, Q)_\eta \times H_{2m}(M, Q)_\eta$ of the mid-dimensional rational intersection form for M; $i_\eta(M)$ is the index of λ_η; $i(M)$ is the index of M.

THEOREM 2.1. *Suppose* codimension$_M(K) \geqq 4$; $p = 4q + 1$ *with* $q = odd$. *Then there are these relations between* $r(K)$, det(K), $i(M)$, $i_\eta(M)$:

(a) *if* $\dim(K) = 0(4)$;

$r(K)$	det(K)	$i_\eta(M)$ mod 8
1	0	$(p-1)i(M) + 4$
1	1	$(p-1)i(M)$
0	1	$(p-1)i(M) + 4$
0	0	$(p-1)i(M)$

(b) *if* $\dim(K) = 2(4)$; $i_\eta(M) = (p-1)i(M)$ mod 8.

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