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# COMPACTNESS IN LOCALLY COMPACT GROUPS[1]

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In [**4**], Glicksberg proved that every $\omega(G, \hat{G})$-compact subset of a LCA group $(G, \mathscr{T})$ is $\mathscr{T}$-compact. In this note three generalizations of this result are given (viz., Theorem 1, Theorem 3 and Theorem 4). In each case, a set is given two comparable topologies whose compacta are shown to coincide. It follows (with the weaker topology $T_2$) that sequential convergence also coincides in the two topologies.

THEOREM 1. *Let $(G, \mathscr{T})$ be a locally compact $T_2$ group (not necessarily abelian) with $\hat{G}$ the continuous irreducible unitary representations of G. Then every $\omega(G, \hat{G})$-compact subset of G is $\mathscr{T}$-compact. (For each $\pi$ in $\hat{G}$, the set of unitary operators on $H^\pi$ is given the weak operator topology.)*

The separable metric case of Theorem 1 is proved first. A countable separating subfamily of $\hat{G}$ is shown to exist and an argument of Eberlein used to prove that every weakly compact set is weakly sequentially compact. Ernest proved [**3**, Corollary 4.5] that every weakly convergent sequence is $\mathscr{T}$-convergent, so this case of the proof is complete, since $\mathscr{T}$ is metric. The $\sigma$-compact case is shown to follow by virtue of the fact that a compact normal subgroup can be factored out to leave a separable metric quotient group. The general locally compact $T_2$ case is then established by showing (via irreducible positive definite functions) that every weakly compact set must lie in an open $\sigma$-compact subgroup of $G$.

The following theorem settles the question raised and partially answered by Bichteler in [**1**]. It is an immediate consequence of Theorem 1.

THEOREM 2. *Let $\mathscr{T}_1$, $\mathscr{T}_2$ be locally compact Hausdorff topologies on a group G, which give rise to the same continuous irreducible unitary representations of G. Then $\mathscr{T}_1 = \mathscr{T}_2$.*

THEOREM 3. *Let $(G, \mathscr{T})$ be a locally compact $T_2$ group and let $P(G)$ be the set of continuous positive definite functions on G. Then every subset of $P(G)$ compact in the topology of pointwise convergence is compact in the compact-open topology.*

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COROLLARY. *Let $X$ be a topological space such that $X \times G$ is a k-space, and let $f: X \times G \to C$ be a separately continuous function which is positive definite in the second variable. Then $f$ is jointly continuous.*

(Note that Glicksberg's theorem mentioned above follows from Theorem 3 since, for $G$ abelian, $G \subseteq P(\hat{G})$.)

The next result, published by Corson and Glicksberg in [**2**], was obtained independently by us in 1968, with a somewhat different proof.

THEOREM 4. *Let $G$ and $H$ be topological groups and suppose the closure of every compactly generated subgroup of $G$ is of second category in itself. Then every subset of the set* $\mathrm{Hom}(G, H)$ *(of continuous homomorphisms on $G$ to $H$) compact in the topology of pointwise convergence is compact in the compact-open topology.*

COROLLARY. *Let $X$ be a topological space such that $X \times G$ is a k-space and let $f: X \times G \to H$ be a separately continuous function such that $f(x, g)$ is a homomorphism in $g$, for each $x \in X$. Then $f$ is jointly continuous.*

In the proofs of both Theorem 3 and Theorem 4, a well-known theorem of Grothendieck [**5**, Theorem 5] on the equivalence of compactness and sequential compactness in certain function spaces is combined with a modified version of the Arzela-Ascoli theorem to allow consideration of sequential convergence. In Theorem 3, Haar measure and the Lebesgue Dominated Convergence Theorem are utilized, whereas Theorem 4 depends on a category theorem of Osgood [**6**, Theorem 9.5].

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