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CURVATURE MEASURES FOR PIECEWISE LINEAR MANIFOLDS[1]

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Let K be a convex cell of dimension m in Euclidean n-space, R^n. The volume of the tubular neighborhood of radius ρ around K is given by a polynomial, in ρ,

$$\sum_p \sum_i H^p(K_p^i) \frac{v_{n-m}}{v_{m-p}} \frac{\rho^{n-p}}{n-p} \int_{c_p^i} dS^{m-p-1},$$

where H^p is the p-dimensional Hausdorff measure in R^n, dS^k is the volume element of the standard unit sphere in R^k, v_k is $H^{k-1}(S^{k-1})$, K_p^i is a face of dimension p, c_p^i is the outer normal angle determined by K_p^i, p varies from 0 to m, i varies from 1 to N_p = the number of faces of dimension p, and $m < n$.

From this formula we can define the *pth curvature measure* of K as follows. For any bounded Borel set $A \subset R^n$,

$$\sigma_p(A) = \sum_i H^p(A \cap K_p^i) \frac{1}{v_{m-p}} \int_{c_p^i} dS^{m-p-1}.$$

In addition to being measures, the σ_p are invariant under the full Euclidean group of rigid motions in R^n and satisfy the following strong stability property.

THEOREM 1. *Let L be a k-dimensional affine subspace of R^n and $\xi(n, k)$ the volume element of the manifold $E(n, k)$ of all k-dimensional affine subspaces in R^n. Then*

$$\int_{L \cap K \neq \varnothing} \sigma_j(L \cap K)\xi(n, k) = c_j \sigma_{n-k+j}(K),$$

where c_j is a constant depending on n, m, k.

Given a piecewise linear manifold K of dimension m, with boundary ∂K, piecewise linearly embedded in R^n one can also define the pth curvature measure of K. For any bounded Borel set $A \subset R^n$,

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$$\sigma_p(A) = \sum H^p(A \cap s_p) \frac{1}{v_{m-p}} \int_{\bar{s}_p} dS^{m-p-1}$$

where the summation is over all cells s_p in the p-skeleton of K and where $\bar{s}_p = \sum_k \sum_j (-1)^k \bar{c}_{k,j}$ with k running from 0 to p and j running over all interior angles in a convex decomposition of the star of s_p; moreover the $\bar{s}_p$ are chains on S^{m-p-1}.

Note that the supports of the σ_p are contained in ∂K for $p \leqq m - 1 = n - 1$ since K is a manifold, and that the curvature measures are now Radon measures (in the sense of Bourbaki).

THEOREM 2. *The curvature measures σ_p of a piecewise linear submanifold K of R^n are invariant under the group of rigid motions of R^n and are stable in the sense of Theorem* 1.

A Radon measure ϕ on R^n is called a geometric measure of dimension m on R^n iff ϕ is a real linear combination of Radon measures $\phi_j, j = 0, 1, \ldots, m$ for which the following conditions hold: $\phi_j(A) = 0$ if the dimension of A is less than j, ϕ_j is rigid motion invariant, ϕ_0 is a topological invariant and

$$\int_{L \cap K \neq \varnothing} \phi_j(K \cap L)\xi(n, k) = b_j \phi_{n-k+j}(K) \qquad (n - j \leqq m),$$

where the notation is the same as in Theorem 1, K a piecewise linear submanifold of R^n and b_j is a constant depending only on dimensional considerations.

THEOREM 3. *Let ϕ be a geometric measure of dimension m on R^n, then there exist real numbers a_i such that*

$$\phi = \sum_i a_i \sigma_i, \qquad i = 0, 1, \ldots, m,$$

in the sense that, for any piecewise linear submanifold K of R^n,

$$\phi(K) = \sum_i a_i \sigma_i(K).$$

Blaschke has proved the curvature measures of simplicial complexes generate the space of finitely additive rigid motion invariant set functions ϕ that are bounded and for which ϕ_3 is $SL(3, R)$ invariant [**1**]. Hadwiger has proved that the curvature measures of convex sets generate the space of finitely additive rigid motion invariant set functions that are continuous with respect to Hausdorff distance [**4**]. Our definition of the curvature measures extends these measures to sets that may not have positive reach in the sense of Federer [**3**]. The curvature measures can, of course, be defined in the smooth case [2] and together with the techniques given here, the curvature measures may also be defined for piecewise differentiable manifolds.

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